AN ELEMENTARY TREATISE ON THE DIFFERENTIAL AND INTEGRAL CALCULUS.

BY THE

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PREFACE.

ANALYTICAL science, after having been long neglected in these countries as an elementary department of education, has, within a few years, been cultivated by the young aspirants for mathematical celebrity with an ardour, and prosecuted with a rapidity and success, which its warmest admirers could scarcely have hoped for. This change would probably have taken place at an earlier period, but for the obstacle opposed to it by the want of treatises on the subject, in our language, of a sufficiently elementary nature. The restless activity of the human mind in the pursuit of knowledge was not long to be checked by so trifling an impediment, and our students soon found in foreign works that which our own professors had failed to supply; and through the medium of these treatises, analytical science began, and has continued, to be cultivated at the universities with singular success. In the mean time, several original works have appeared, which

are gradually superseding the works of foreign professors. For these, the public are indebted to some of the distinguished members of the University of Cambridge; Woodhouse, Whewell, Herschell, Babbage, Peacock, &c., &c.

Desirous of contributing to the great work of improvement which was thus in progress, I some time since published the first part of a treatise on Analytic Geometry; a subject which had not then, nor has been yet treated of by any other English author. The favourable manner in which that work has been received has encouraged me in the prosecution of my labours, the result of which I now venture to offer to the public.

The present Treatise is divided into four parts, the subjects of which are severally, 1. the Differential Calculus; 2. The Integral Calculus; 3. The Calculus of Variations; 4. The Calculus of Differences. The arrangement which I have adopted throughout the work has been to present to the student theory and illustration in alternate sections. I have found by repeated experience, that as on the one hand, the total omission of examples, so common in foreign treatises, renders the theory obscure and even unintelligible; so, on the other hand, their too frequent

recurrence in the progress of the development of the abstract principles of a science is apt to break the continuity and oneness of the reasoning, and to render it difficult for the student to take a general view of the subject as a whole.

By the method adopted in this work, I have attempted to remove both these defects. The student will find in general, that the complete theory of each department of the subject is fully explained before the current of his ideas is stopped by an example. At the same time the subdivisions of the subject are so numerous, and the sections of illustration so frequent, that none of the confusion which is apt to arise from a long exposition of abstract principles without examples of their application can ensue.

Another advantage of this method is, that it is suited to students of different capacities. The sections of illustration will receive only that degree of attention which is necessary to fix clearly in the mind the general principles which have been established in the preceding sections.

There is one part of the work, the calculus of differences, which I am sensible of having written under considerable disadvantage. The treatise on this sub-

ject by Mr. Herschell, which forms the appendix to the translation of the Calculus of Lacroix, together with the collection of examples by the same author, which accompanies Mr. Peacock's examples on the differential and integral calculus, form a treatise on the calculus of differences so excellent, that it would be useless as well as presumptuous in me to attempt to improve it. Under these circumstances, I have confined myself in the fourth part to a few of the most elementary and generally useful principles of differences, particularly those connected with the method of interpolation and the summation of series.

I have attempted in this treatise to include a more extensive range of analytical science, more fully developed, accompanied by a greater quantity of practical illustration, under a considerably less bulk than will be found in most of the foreign treatises on the same subject, particularly those which have hitherto formed the class books at the universities. Whether I have succeeded in this design, I leave the public to decide.

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PART I.

THE DIFFERENTIAL CALCULUS.

THE

ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

PART I.

THE DIFFERENTIAL CALCULUS.

SECTION I.

Preliminary Principles.

(1.) QUANTITIES engaged in this science are considered as *constant* or *variable.*

A quantity, which is supposed to retain the same value throughout the whole of any investigation, is said to be *constant.* On the contrary, a quantity to which in any investigation different values may successively be ascribed, is said to be *variable.*

Constant quantities are usually expressed by the first letters of the alphabet, and variable quantities by the last. Constant and variable quantities are not, however, analogous to *known* and *unknown* quantities in common algebra, since a constant quantity may be unknown.

(2.) The following may serve as examples of constant and variable quantities. A point being given within a circle given in magnitude and position, a line drawn from the given point to the circumference of the circle is in general a variable quantity, as its length will change with the point

in the circumference to which it is drawn. But if the given point within the circle be the centre, the same line becomes a constant quantity, being the same length to whatever point in the circle it is supposed to be drawn. Again, if the base and vertical angle of a triangle be given, the radius of the inscribed circle, and the distance of its centre from the vertex, are variable quantities; but the radius of the circumscribed circle, and the distance of its centre from the vertex, are constant quantities.

(3.) When two variable quantities enter the same investigation, they are frequently so related that the variation of either may be determined by that of the other. In other words, a relation may subsist between them, such, that any *particular* value being assigned to either, the corresponding value of the other will be determined. In this case, either of the variables is said to be a *function* of the other. Thus, for example, in the equation $u = 4 \sin. x$, any variation in x produces a corresponding variation in u, and *vice versa*. Also, any particular value, as 30°, being assigned to x, the corresponding value (2.) of u is determined. In this case, therefore, u is said to be a function of x, or x a function of u indifferently. The same may be observed of the equations $u = 10x^4$, $u = \log. x$, $u = a^x$, &c.

(4.) As it is necessary to express functions without regard to any particular form, a peculiar notation has been invented for this purpose. The character F or f signifies a function, and $\text{F}(x)$ or $f(x)$ signifies a function of x, x being considered the variable. Thus, $u = \text{F}(x)$ signifies that u is a function of x *.

If the variable x be supposed successively to assume all values from zero to infinity, the function $\text{F}(x)$ or u assumes

* The characters $\phi(x)$ and $\psi(x)$, and others, are also used to express functions.

a succession of corresponding values. The *rate* of the variation of u compared with that of x *in general* will change with the value of x. There is but one case in which their rates of variation will have an invariable ratio, which is when $u = ax$, a being a constant quantity. In this case it is obvious that u varies as x. The immediate object of the Differential Calculus is to determine the *rate* of variation of a function relatively to that of its variable.

(5.) It was nearly under this point of view that NEWTON presented the first principles of the Fluxional Calculus. He considers quantities to be generated by motion, as lines are produced by the motion of a point, surfaces by that of a line, &c. The quantity thus *flowing* or varying he called a *fluent*, and the *rate* or *velocity* of its increase or decrease he called its *fluxion*. The Fluxional Calculus was therefore a method of determining the velocity with which a function varies at any point of time compared with the velocity with with its variable changes.

(6.) The conceptions of *motion* and *time*, which are involved in this method, were considered inconsistent with the rigour of mathematical reasoning, and wholly foreign to that science. As an improvement upon the principle, D'ALEMBERT proposed the method of limits. Considering u as a function of x, let the variable x be supposed to receive any finite increment h, so that it becomes $x + h$, and let the corresponding value of u be u', so that we shall have the equations

$$u = \text{F}(x),$$
$$u' = \text{F}(x + h).$$

Let the value of $\frac{u'-u}{h}$ be found. This will be in general a quantity whose value will depend on those of x and h, and it will express the ratio of the finite increment $(u' - u)$ of the function, to (h) that of the variable. If in this quantity

h be supposed to be $= 0$, it will express the *limit of the ratio* * of the corresponding variations of the function and variable, these variations being reduced to infinite minuteness. It is not difficult to perceive that this method attains the same end as the former; but in rejecting the mechanical ideas of time and motion introduces those quantities or increments infinitely minute.

(7.) The last improvement in the principles on which the calculus is founded is that of LAGRANGE. He equally rejects the limits of the ratios of D'Alembert and the motions and velocities of Newton, and has proposed fundamental principles for the calculus at once rigorously demonstrable and purely analytical. Let $u = \mathrm{F}(x)$ and $u' = \mathrm{F}(x + h)$. By developing $u' - u$ in a series of ascending integral and positive powers of h (which may be proved when x is variable to be always possible), let the series

$$u' - u = \mathrm{A}'h + \mathrm{A}''h^2 + \mathrm{A}'''h^3 \quad . \quad . \quad . \quad .$$

be obtained.

In this series the coefficients A', A'', A''', &c. are functions of x. The function A' is called by Lagrange the *first derived function*. This may be shown to be the same quantity which D'Alembert calls the limit of the ratio of the corresponding increments of the function and variable. Let both members of the preceding equation be divided by h,

$$\frac{u' - u}{h} = \mathrm{A}' + \mathrm{A}''h + \mathrm{A}'''h^2. \quad . \quad . \quad . \quad .$$

If $h = 0$ this becomes A', which is therefore the limit of the ratio.

(8.) Thus these three methods of presenting the first

* A *limit* is a state to which a quantity *continually* approaches, and nearer to which it comes than any assignable difference, but to which it cannot actually attain.

principles of the calculus to the student arrive at the same end, though by different means.

Newton proposed to determine the ratio of the velocities with which the function and variable increase or decrease, and called these velocities their *fluxions*. The notation by which he expressed the fluxions was, $\dot{u}$, $\dot{x}$, the function u and its variable x being called *fluents*. The quantity $\frac{\dot{u}}{\dot{x}}$ is the *fluxional coefficient*. It may be observed here that the fluxions are not quantities absolutely determinate, but may have any values, provided that their ratio is that of the velocities with which the function and variable change. The *fluxional coefficient*, however, is given for any particular value of x, and, in general, only varies with x.

D'Alembert proposed to determine the value of the fraction having for its numerator and denominator the simultaneous increments of the function and variable, when both these increments are $= 0$. The value thus determined is called *the differential coefficient*, and two indeterminate quantities, du and dx being assumed, so that the fraction $\frac{du}{dx}$ shall have this value, are called *the differential* (du) of the function, and *the differential* (dx) of the variable. The notation du, dx, is not meant to express $d \times u$, $d \times x$, but simply "the differential of u" and "the differential of x." It is evident that the "differentials" of the function and variable, according to this system, are the same quantities as the "fluxions" in the Newtonian method, differing only in notation and name.

Lagrange attempted to set aside both the notation and nomenclature of the differential and fluxional calculus. He showed that the true principles of this science consisted in the methods of developing functions in series, and were altogether independent of the ideas of velocities or of infinitesimal

or evanescent quantities, or even of the limits of ratios. He proved that if in any function u of a variable x, the variable be supposed to be changed to $x + h$, the function $\text{F}(x + h)$ or u' could be always expanded in a series of ascending integral and positive powers of h, provided that the variable *x is not supposed to have any particular value.* If this development be

$$u' = \text{A} + \text{A}'h + \text{A}''h^2 + \text{A}'''h^3 + \text{\&c.} \quad . \quad . \quad .$$

he called the coefficient A' of the second term, *the first derived function* of the function u. From what has been already observed, it appears that this is the same as the "fluxional coefficient" of Newton, and the "limit of the ratio" or differential coefficient of D'Alembert. It is also evident, that the second term of the series is the differential of the function u, h being assumed as dx *.

(9.) We shall in the following treatise adopt the notation of the differential calculus in preference to that of the fluxional, as well because it is generally received by the scientific world at present, as because of its superior simplicity and power. We shall, however, use the principles of all the three methods as they may seem best suited to the subject of investigation †.

(10.) Functions are *explicit* or *implicit*.

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* In this enumeration of the methods of the different founders of the calculus, I have omitted Leibnitz's infinitesimal method, because, although I believe it was the first promulged and published, yet it is inferior in rigour to the others. Its validity consists in a kind of compensation of errors.

† Wherever it can be used without too great complexity for so elementary a treatise as the present, I have preferred the method of Lagrange, as being most rigorous, and free from metaphysical objections.

An explicit function is one whose form is known. Thus, x^m, log. x, sin. x, a^x, are explicit functions of x.

An implicit function is one whose form is unknown, or at least not expressed. Thus, if $u^3 + u^2ax + ux + 1 = 0$, u is an implicit function of x, being a root of this equation. Also, if $u = \sin. y$, and $yx^3 + bx^2 + cx + d = 0$, u is an implicit function of x; b, c, and d being supposed to be constant quantities. The roots of an equation are implicit functions of its coefficients.

Functions also are of one or several variables. If $u=x^m$, u is a function of one variable, m being supposed constant. If $u = x^y$, u is a function of two variables x and y.

Again, if $u = x^{\frac{y}{z}}$, u is a function of three variables, and so on. In these cases the variables are supposed to be *independent*, that is, the variation of either or any one of them is independent of the others, which, at the same time, may or may not be varied. If, however, any two of the variables be connected by any equation or condition, they cease to be independent variables, as any change in either produces a corresponding change in the other. Thus, if $u = x^y$, and at the same time $x = 2y$, x and y are not independent variables, and the function in this case, though apparently a function of two variables, is *implicitly* a function of one variable, and becomes an explicit function of one variable by eliminating y, whereby $u = x^{\frac{1}{2}x}$.

(11.) Functions are also divided with respect to their form into *algebraic* and *transcendental*. Those in which the variable is united with the constants by common algebraical operations, are called *algebraic functions*. Such are ax, $\frac{a}{x}$, x^a, $\sqrt[a]{x}$, &c. But those in which the variable is connected otherwise with the constants, are called *transcendental functions*. Such are x^x, a log. x, a sin. x, &c.

The process by which the differential of a function is found, is called "differentiation," and the function is said to be "differentiated."

We shall commence by explaining the methods of differentiating functions, whether explicit or implicit of a single variable.

SECTION II.

The differentiation of functions of one variable.

PROP. I.

(12.) *If $u = \text{F}(x)$ and x be changed into $x + h$, so that $u' = \text{F}(x + h)$ u' may be developed in a series of positive and integral powers of h, provided that x be an indeterminate quantity.*

Let

$$u' = \text{A}h^a + \text{B}h^b + \text{C}h^c \quad . \quad . \quad . \quad .$$

the quantities a, b, c . . . must be positive and integral, for

1°. If any of these exponents were negative, the supposition $h = 0$ would render u' (which then becomes equal to u) infinite. Hence x must have that *determinate* value which renders $\dfrac{1}{\text{F}(x)} = 0$, which is contrary to hypothesis.

Also, since when $h = 0$, $u' = u$, it follows that one of the terms of the series must be independent of h, and that the value of that term must be $\text{F}(x)$. Hence the series must be of the form

$$u' = \text{F}(x) + \text{A}h^a + \text{B}h^b + \text{C}h^c. \quad . \quad . \quad . \quad . \quad .$$

2°. If any of the exponents were fractional, there would be as many values of the term which involved that power as there were units in the denominator of the fractional exponent. Now it is plain, that the radicals affecting h can only arise from radicals included in the primitive function $F(x)$, and that the substitution of $x + h$ for x can neither increase nor diminish the number of these radicals, nor change their nature, so long as x and h remain indeterminate. On the other hand, it appears from the theory of equations, that every radical has as many different values as there are units in its exponent, and that every irrational function has consequently as many different values as there are different combinations of the values of the radicals which it includes. Therefore, if the development of $F(x + h)$ could contain a term of the form $gh^{\frac{m}{n}}$, the function $F(x)$ must necessarily be irrational, and must have consequently a certain number of different values, and therefore $F(x + h)$ must have the same number; but the development of this last in a series being

$$F(x + h) = F(x) + Ah^{a} + Bh^{b} \ldots . gh^{\frac{m}{n}} \ldots .$$

each value of $F(x)$ is successively combined with the n values of $g\sqrt[n]{h^m}$, so that the function $F(x + h)$ has a greater number of values when developed, than it has when not developed, which is absurd *. Hence no power of h can occur in the development, except such as have positive integers as exponents. The series must therefore have the form,

$$F(x + h) = F(x) + A'h + A''h^2 + A'''h^3 \ldots .$$

(13.) *Cor.* Hence

$$u' - u = A'h + A''h^2 + A'''h^3 \ldots .$$

By dividing both sides by h, and supposing $h = 0$, it appears that the coefficient A' of h is the limit of the ratio of

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* Theorie des Fonctions Analytique. LAGRANGE, p. 7.

the increment $u' - u$ of the function to the corresponding increment h of the variable. This is therefore the differential coefficient, and $\frac{du}{dx} = A'$.

(14.) *Cor.* 2. As h^2 is a common factor of the terms of the series after the first, the series may be expressed thus,

$$u' - u = A'h + sh^2,$$

$$\text{where } s = A'' + A'''h + A''''h^2 \ldots\ldots$$

(15.) *Cor.* 3. The first term $A'h$ of the expanded difference $u' - u$ of the function may always be considered as its differential.

PROP. II.

(16.) *If* $u = F(y)$ *and* $y = f(x)$ *to determine the differential coefficient of* u *considered as an implicit function of* x.

Let $y' = f(x + h)$, and

$$y' - y = A'h + sh^2.$$

If $y' - y = k$, and $\because y' = y + k$, $u' = F(y + k)$,

$$u' - u = B'k + s'k^2,$$

substituting for k in this its value given by the former, the result arranged by the dimensions of h will be of the form

$$u' - u = A'B'h + s''h^2.$$

By these three series we find

$$\frac{dy}{dx} = A',$$

$$\frac{du}{dy} = B',$$

$$\frac{du}{dx} = A'B'.$$

From which it follows, that

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}.$$

Therefore, if there be three quantities u, y, x, each a function of the other, the differential coefficient of any one u considered as a function of another x is equal to the product of the differential coefficients of that one u considered as a function of y, and of y considered as a function of the remaining one x.

It is obvious that by continuing the process, the same principle may be shown to be applicable to any number of differential coefficients.

PROP. III.

(17.) *To differentiate a quantity which is composed of several functions of the same variable united by addition or subtraction, the differentials of the component functions being given.*

Let $u = v + y - z$, v, y, and z being functions of the same variable x. Let

$$v' - v = \text{A}h + \text{B}h^2,$$
$$y' - y = \text{A}'h + \text{B}'h^2,$$
$$z' - z = \text{A}''h + \text{B}''h^2.$$

Adding the first two and subtracting the third, observing the conditions,

$$u = v + y - z,$$
$$u' = v' + y' - z',$$

the result will be

$$u' - u = \text{A}h + \text{A}'h - \text{A}''h + (\text{B} + \text{B}' - \text{B}'')h^2.$$

Hence

$$du = dv + dy - dz.$$

Hence it is clear that the result is in general the differentials of the several functions united together in the same manner, and with the same signs as the functions themselves.

PROP. IV.

(18.) *Constant quantities combined with a function by addition or subtraction disappear in its differential, and all constant quantities which are combined with it as factors are similarly combined with its differential.*

1°. Let $u = \text{F}(x) \pm a$, $\therefore u' = \text{F}(x+h) \pm a$, $\therefore$

$$\frac{u'-u}{h} = \frac{\text{F}(x+h)-\text{F}(x)}{h}.$$

In which a does not appear, and therefore it does not appear in the differential of u, which is deduced from this.

Hence it follows that u has the same differential, whatever the constant a may be.

2°. Let $u = a\text{F}(x)$, $\therefore u' = a\text{F}(x+h)$, $\therefore$

$$u' - u = a[\text{F}(x+h) - \text{F}(x)],$$

$$\therefore \frac{u'-u}{h} = a \cdot \frac{\text{F}(x+h)-\text{F}(x)}{h},$$

from which the differential coefficient being derived, it is evident that a is a factor of it.

The same observation obviously applies to constant divisors, since a may be $\frac{1}{b}$.

(19.) *Def.* Functions of the form $u = a^x$ are called *exponential functions.*

PROP. V.

(20.) *To differentiate an exponential function.*

Let $u = a^x$, $\therefore u' = a^{(x+h)} = a^x \cdot a^h$. Let $a = 1 + b$, $\therefore a^h = (1+b)^h$. This being expanded by the binomial theorem, gives

$$a^h = 1 + \frac{h}{1} \cdot b + \frac{h \cdot h-1}{1 \cdot 2} b^2 + \frac{h \cdot h-1 \cdot h-2}{1 \cdot 2 \cdot 3} \cdot b^3 + \ldots$$

which arranged by the dimensions of h, is of the form

$$a^h = 1 + kh + sh^2,$$

where

$$k = \frac{b}{1} - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} \dots\dots$$

multiplying both by a^x, and substituting u for a^x, and u' for $a^x a^h$, the result after transposing u is

$$u' - u = kuh + suh^2.$$

Hence $du = kudx = ka^x dx$, and $\therefore \frac{du}{dx} = ku = ka^x$.

The value of the series k will be determined in finite terms hereafter (64.).

PROP. VI.

(21.) *To differentiate a logarithm.*

Assuming the logarithms of the equation $u = a^x$ relatively to the base a, we find $lu = x$, $\therefore d \cdot lu = dx$. Eliminating dx by this, and the equation $du = kudx$ found in the last proposition, we find

$$d \cdot lu = \frac{1}{k} \cdot \frac{du}{u}.$$

It is obvious that the value of k depends upon that of the base a. The base which renders $k = 1$ is called the *Neperian base,* and sometimes the *hyperbolic base;* and the corresponding logarithms are called *Neperian* or *hyperbolic logarithms* *. The value of the Neperian base will be determined hereafter (64.).

The quantity $\frac{1}{k}$ is called the modulus of the system, whose base is a.

Hence the modulus of hyperbolic logarithms is unity.

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* For the origin of the term *hyperbolic logarithm,* see my Algebraic Geometry, Art. (385.).

Logarithms of the hyperbolic system are sometimes denoted thus, l'; and those related to any other base a, thus, l or L *.

Hence $d \cdot l'u = \frac{du}{u}$, and, in general, if m be the modulus $d \cdot lu = m \cdot \frac{du}{u}$.

PROP. VII.

(22.) *To differentiate a function which is the product of several functions of the same variable.*

Let $u = y'y''y''' \ldots y^{(n)}$, n being the number of factors, and the factors being all functions of x. Assuming the hyperbolic logarithms,

$$l'u = l'y' + l'y'' + l'y''' \ldots l'y^{(n)}.$$

Differentiating this (21.),

$$\frac{du}{u} = \frac{dy'}{y'} + \frac{dy''}{y''} + \frac{dy'''}{y'''} \ldots \frac{dy^{(n)}}{y^{(n)}}.$$

Multiplying this by the original equation, the result is

$$du = y''y''' \ldots y^{(n)} \cdot dy' + y'y''' \ldots y^{(n)}dy'' + \ldots y'y''y''' \ldots y^{(n-1)}\,dy^{(n)}.$$

Therefore "the differential of the product of several functions is equal to the sum of the products formed by multiplying the differential of each function by the product of the remaining functions." Thus, if

$$u = y'y'',$$
$$du = y'dy'' + y''dy';$$

that is, "the differential of the product of two functions is equal to the sum of the products of the functions into their alternate differentials."

If $u = y'y''y'''$,

$$du = y'y''dy''' + y'y'''dy'' + y''y'''dy'.$$

* We shall generally use the Neperian logarithms without any distinguishing mark. Whenever any other logarithm is used it will be expressly mentioned.

PROP. VIII.

(23.) *To differentiate a fraction whose numerator and denominator are each products of several functions of the same variable.*

Let $u = \dfrac{y'y''y''' \dots y^{(n)}}{z'z''z''' \dots z^{(m)}}$, where $y', y'', \dots z', z'', \dots$ are functions of x. Assuming the logarithms,

$$lu = ly' + ly'' + \dots ly^{(n)} - lz' - lz'' - \dots - lz^{(m)},$$

$$\therefore \frac{du}{u} = \frac{dy'}{y'} + \frac{dy''}{y''} + \dots - \frac{dz'}{z'} - \frac{dz''}{z''}.$$

By multiplying this by the original equation, the value of du can be found, $\therefore$

$$du = u\left\{\frac{dy'}{y'} + \frac{dy''}{y''} \dots - \frac{dz'}{z'} - \frac{dz''}{z''} \dots\right\}$$

The differential of a fraction is therefore equal to the product found by multiplying the fraction itself into the sum of all the differentials of the functions in the numerator, divided by the functions respectively, diminished by the sum of the differentials of all the functions in the denominator divided by the functions respectively.

Thus, if $u = \dfrac{y'}{z'}$,

$$du = u\left(\frac{dy'}{y'} - \frac{dz'}{z'}\right) = \frac{z'dy' - y'dz'}{z'^2}.$$

PROP. IX.

(24.) *To differentiate a power.*

Let $u = x^m$. Assuming the logarithms $lu = mlx$. Hence

$$\frac{du}{u} = m\frac{dx}{x},$$

$$\therefore du = mx^{m-1}dx.$$

It may be observed, that this is perfectly general. The exponent m may be positive or negative, integral or fractional.

LEMMA I.

(25.) *The limit of the ratios of the chord of a circular arc, and the arc itself to the tangent, the arc being diminished without limit, is a ratio of equality.*

Let the arc be x related to the radius unity, by trigonometry.

$$\frac{\text{chord. } x}{\tan. x} = \frac{2 \sin. \frac{1}{2}x}{\tan. x} = \frac{2 \sin. \frac{1}{2}x \cdot \cos. x}{\sin. x};$$

but $\sin. x = 2 \sin. \frac{1}{2}x \cos. \frac{1}{2}x$,

$$\therefore \frac{\text{chord. } x}{\tan. x} = \frac{\cos. x}{\cos. \frac{1}{2}x}.$$

In the limit, when $x = 0$ and $\therefore \frac{1}{2}x = 0$, $\cos. x = \cos \frac{1}{2}x = 1$, whence the limit of the ratio of the chord to the tangent is a ratio of equality.

Since the arc is included between the chord and tangent, it is evident that the limit of the ratio of it to either is a ratio of equality.

LEMMA II.

(26.) *The limit of the ratio of the sine of an arc to the arc itself, both being infinitely diminished, is a ratio of equality.*

Let the arc be x related to the radius unity.

$$\cos. x = \frac{\sin. x}{\tan. x}.$$

If $x = 0$, $\therefore \cos. x = 1$, $\therefore$ the limit of the ratio $\sin. x$, to $\tan. x$, is a ratio of equality. But since the limit of the ratio of the arc to the tangent is a ratio of equality (25), it fol-

lows that the limit of the ratio of the sine to the arc is a ratio of equality.

PROP. X.

(27.) *To differentiate the sine of an arc, considered as a function of the arc itself.*

Let $u = \sin. x$, and $u' = \sin. (x + h)$, $\therefore$

$u' - u = \sin. (x + h) - \sin. x = 2 \sin. \frac{1}{2}h \cos. (x + \frac{1}{2}h)$,

$$\therefore \frac{u' - u}{h} = \frac{\sin. \frac{1}{2}h}{\frac{1}{2}h} \cdot \cos. (x + \frac{1}{2}h).$$

If $h = 0$, by (26), $\frac{\sin \frac{1}{2}h}{\frac{1}{2}h} = 1$, and $\cos. (x + \frac{1}{2}h) = \cos. x$, hence

$$\frac{du}{dx} = \cos. x,$$

$$\therefore du = \cos. x dx.$$

PROP. XI.

(28.) *To differentiate the cosine of an arc, considered as a function of the arc itself.*

Let $u = \cos. x$, $\therefore u = \sin. (\frac{\pi}{2} - x)$, $\therefore$ by (27.),

$$du = \cos. (\frac{\pi}{2} - x) d (\frac{\pi}{2} - x).$$

But $\frac{\pi}{2}$ being constant, has no differential, and

$$\cos. (\frac{\pi}{2} - x) = \sin x, \text{ therefore}$$

$$du = - \sin. x dx.$$

PROP. XII.

(29.) *To differentiate the tangent and cotangent, considered as functions of the arc.*

1°. Let $u = \tan. x = \frac{\sin. x}{\cos. x}$, $\therefore$ by (23.),

$$du = \frac{\cos.x d \sin.x - \sin.x d \cos.x}{\cos.^2 x};$$

but $d \sin.x = \cos.x dx$, and $d \cos.x = - \sin.x dx$. Making these substitutions, and observing the condition,

$$\sin.^2 x + \cos.^2 x = 1,$$

the result is

$$du = \frac{dx}{\cos.^2 x}.$$

2°. Since $\cot. x = \tan. (\frac{\pi}{2} - x)$, it follows from this that

$$d \cot.x = - \frac{dx}{\sin.^2 x}.$$

PROP. XIII.

(30.) *To differentiate the secant and cosecant as functions of the arc.*

1°. Let $u = \sec.x = \frac{1}{\cos. x}$, ∴

$$du = \frac{\sin.x.dx}{\cos.^2 x} = \tan.x \sec.x.dx.$$

2°. Let $u = \operatorname{cosec}.x = \sec. (\frac{\pi}{2} - x)$, ∴

$$du = - \tan. (\frac{\pi}{2} - x) \cdot \sec. (\frac{\pi}{2} - x) dx$$

$$\text{or } du = - \cot.x. \operatorname{cosec}.x. dx.$$

PROP. XIV.

(31.) *To differentiate an arc, considered as a function of its sine or cosine.*

1°. Let $u = \sin.^{-1} x$ *, ∵ $\sin. u = x$, ∴ $\cos. u du = dx$. But $\cos. u = \sqrt{1 - x^2}$, hence

* $\sin.^{-1} x$ signifies the arc whose sine is x.

$$\therefore du = \frac{dx}{\sqrt{1-x^2}}.$$

2°. Let $u = \cos.^{-1} x$. Since $\sin.^{-1} x + \cos.^{-1} x = \frac{\pi}{2}$, $\therefore$

$$d \sin.^{-1} x + d \cos.^{-1} x = 0,$$
$$d \cos.^{-1} x = - d \sin.^{-1} x,$$
$$\therefore d \cos.^{-1} x = - \frac{dx}{\sqrt{1-x^2}}.$$

PROP. XV.

(32.) *To differentiate an arc as a function of its tangent or cotangent.*

1°. Let $u = \tan.^{-1} x$, $\therefore \tan. u = x$, $\therefore \frac{du}{\cos.^2 u} = dx$. But since

$$\cos.^2 u = \frac{1}{\sec.^2 u} = \frac{1}{1+x^2},$$
$$\therefore du = \frac{dx}{1+x^2}.$$

2°. Let $u = \cot.^{-1} x$. Since $\tan.^{-1} x + \cot.^{-1} x = \frac{\pi}{2}$, $\therefore$

$$d \cot.^{-1} x = - d \tan.^{-1} x,$$
$$\therefore du = - \frac{dx}{1+x^2}.$$

PROP. XVI.

(33.) *To differentiate the arc as a function of its secant or cosecant.*

1°. Let $u = \sec.^{-1} x$, $\therefore \sec. u = x$, $\therefore \tan. u . \sec. u . du = dx$. But

$$\sec. u = x,$$
$$\therefore \tan. u = \sqrt{x^2 - 1},$$

$$\therefore du = \frac{dx}{x\sqrt{x^2-1}}.$$

2°. Let $u = \text{cosec.}^{-1} x$. Since $\sec.^{-1} x + \text{cosec.}^{-1} x = \frac{\pi}{2}$,

$$\therefore d\,\text{cosec.}^{-1} x = -\,d\sec.^{-1} x,$$

$$\therefore du = -\frac{dx}{x\sqrt{x^2-1}}.$$

SECTION III.

Praxis on the differentiation of functions of one variable.

Ex. 1. If $u = (a + bx)^2$. Let $z = a + bx$,

$\therefore$ (18.) $\frac{dz}{dx} = b$. But since $u = z^2$ by (24.) $\frac{du}{dz} = 2z$,

and (16.) $\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx}$. Hence $\frac{du}{dx} = 2b(a + bx)$.

Ex. 2. If $u = (a + bx + cx^2)^3$. Let $z = a + bx + cx^2$,

$$\therefore (18.)\ \frac{dz}{dx} = b + 2cx, \text{ and since } u = z^3,$$

$$\therefore (24.)\ \frac{du}{dz} = 3z^2, \text{ hence}$$

$$(16.)\ .\ \frac{du}{dx} = 3(b + 2cx)\,(a + bx + cx^2)^2.$$

Ex. 3. If $u = (a + bx)^m$. As before, let $z = a + bx$,

$\therefore$ $\frac{dz}{dx} = b$, and since $u = z^m$, $\therefore$ (24.) $\frac{du}{dz} = mz^{m-1}$,

$$\therefore (16.)\ \frac{du}{dx} = mb(a + bx)^{m-1}.$$

Ex. 4. If $u = (a + bx)^2\,(a' + b'x)^3$. Let $y = a + bx$, and $y' = a' + b'x$,

$$\therefore u = y^2 y'^3 \cdot \frac{du}{u} = \frac{d(y^2)}{y^2} + \frac{d(y'^3)}{y'^3} = \frac{2dy}{y} + \frac{3dy'}{y'}.$$

But $dy = bdx$, $dy' = b'dx$, $\therefore$

$$\frac{du}{dx} = u\left(\frac{2b}{a+bx} + \frac{3b'}{(a'+b'x)}\right),$$

or $$\frac{du}{dx} = 2b(a+bx)(a'+b'x)^3 + 3b'(a+bx)^2(a'+b'x)^2,$$

or $$\frac{du}{dx} = (a+bx)(a'+b'x)^2\{2b(a'+b'x) + 3b'(a+bx)\}.$$

Ex. 5. If $u = (a+bx)^m(a'+b'x)^{m'}(a''+b''x)^{m''}$.
Let $y = (a+bx)^m$, $y' = (a'+b'x)^{m'}$, $y'' = (a''+b''x)^{m''}$.

$$\text{Hence } du = \left(\frac{dy}{y} + \frac{dy'}{y'} + \frac{dy''}{y''}\right)yy'y'',$$

$$dy = mb(a+bx)^{m-1}dx,$$
$$dy' = m'b'(a'+b'x)^{m'-1}dx,$$
$$dy'' = m''b''(a''+b''x)^{m''-1}dx,$$

$$\therefore \frac{du}{dx} = (a+bx)^m(a'+b'x)^{m'}(a''+b''x)^{m''} \times$$
$$\left\{\frac{mb}{a+bx} + \frac{m'b'}{a'+b'x} + \frac{m''b''}{a''+b''x}\right\}.$$

Ex. 6. If $u = (ax^{m-n}+b)^q$. Let $z = ax^{m-n} + b$,

$$\therefore \frac{dz}{dx} = (m-n)ax^{m-n-1}.$$

Also $u = z^q$, $\therefore \dfrac{du}{dz} = qz^{q-1}$,

$$\therefore \frac{du}{dx} = aq(m-n)x^{m-n-1}(ax^{m-n}+b)^{q-1}.$$

Ex. 7. If $u = \dfrac{a}{x^m}$, $\therefore u = ax^{-m}$, $\therefore \dfrac{du}{dx} = -max^{-m-1}$ by (24.).

Ex. 8. If $u = a\sqrt[n]{x^m}$, $\therefore u = ax^{\frac{m}{n}}$, $\therefore \dfrac{du}{dx} = \dfrac{m}{n}ax^{\frac{m}{n}-1}$ by (24.).

Ex. 9. If $u = \sqrt[n]{a^m - x^m}$. Let $z = a^m - x^m$,

$$\therefore \frac{dz}{dx} = -mx^{m-1}.$$

Also $u = z^{\frac{1}{n}}$, $\because \frac{du}{dz} = \frac{1}{n} z^{\frac{1}{n}-1}$. Hence by (16.),

$$\frac{du}{dx} = -\frac{m}{n} x^{m-1} \cdot (a^m - x^m)^{\frac{1}{n}-1}.$$

Ex. 10. If $u = \frac{1}{\sqrt[n]{a - x^2}}$. Let $z = a - x^2$,

$$\because \frac{dz}{dx} = -2x.$$

Also $u = z^{-\frac{1}{n}}$, $\because \frac{du}{dz} = -\frac{1}{n} \cdot z^{-\frac{1}{n}-1}$, $\because$ (16.),

$$\frac{du}{dx} = \frac{2}{n} \cdot x(a - x^2)^{-\frac{1}{n}-1}.$$

Ex. 11. If $u = \sqrt[4]{\left\{a - \frac{b}{\sqrt{x}} + \sqrt[3]{(c^2 - x^2)}\right\}^3}$. Let

$$\frac{b}{\sqrt{x}} = y;\ \sqrt[3]{c^2 - x^2} = z.$$

Hence we find

$$u = (a - y + z)^{\frac{3}{4}},$$

$$\because du = \tfrac{3}{4}(a - y + z)^{\frac{3}{4}-1}(-dy + dz),$$

$$\text{or } du = \frac{-3dy + 3dz}{4\sqrt[4]{a - y + z}}.$$

But also,

$$dy = d\frac{b}{\sqrt{x}} = -\frac{bdx}{2x\sqrt{x}},$$

$$dz = d(c^2 - x^2)^{\frac{1}{3}} = \tfrac{1}{3}(c^2 - x^2)^{-\frac{2}{3}} \times -2xdx = \frac{-2xdx}{3(c^2 - x^2)^{\frac{2}{3}}}.$$

Hence

$$du = \frac{\frac{3bdx}{2x\sqrt{x}} - \frac{2xdx}{\sqrt[3]{(c^2 - x^2)^2}}}{4\sqrt[4]{a - \frac{b}{\sqrt{x}} + \sqrt[3]{(c^2 - x^2)}}}.$$

Ex. 12. If $u = l' \frac{x}{\sqrt{a^2 + x^2}}$. Let $z = \sqrt{a^2 + x^2}$,

$$\therefore dz = \frac{xdx}{\sqrt{a^2+x^2}}.$$

But $u = l\dfrac{x}{z} = lx - l'z$, $\therefore du = \dfrac{dx}{x} - \dfrac{dz}{z}$. Hence

$$du = \frac{dx}{x} - \frac{xdx}{a^2+x^2} = \frac{a^2dx}{x(a^2+x^2)},$$

$$\therefore \frac{du}{dx} = \frac{a^2}{x(a^2+x^2)}.$$

Ex. 13. If $u = l\{(a+x)^m (a'+x)^{m'}(a''+x)^{m''}\}$.

Hence $u = ml'(a+x) + m'l'(a'+x) + m''l'(a''+x)$,

$$\therefore \frac{du}{dx} = \frac{m}{a+x} + \frac{m'}{a'+x} + \frac{m''}{a''+x},$$

or $\dfrac{du}{dx} =$

$$\frac{(m+m'+m'')x^2 + [m(a'+a'')+m'(a+a'')+m''(a+a')]x + ma'a'' + m'aa'' + m''aa'}{(a+x)(a'+x)(a''+x)}$$

This differential coefficient is evidently of the form

$$\frac{\text{A}x^2+\text{B}x+\text{C}}{x^3+\text{A}'x^2+\text{B}'x+\text{C}'},$$

$-a$, $-a'$, $-a''$, being the roots of the equation

$$x^3 + \text{A}'x^2 + \text{B}'x + \text{C}' = 0.$$

This circumstance is attended with some important consequences in the integral calculus.

Ex. 14. If $u = l'\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$. Let $y = \sqrt{1+x}$,

$$z = \sqrt{1-x},$$

$$u = l'\frac{y+z}{y-z} = l'(y+z) - l'(y-z),$$

$$\therefore du = \frac{dy+dz}{y+z} - \frac{dy-dz}{y-z},$$

$$\therefore du = 2\cdot\frac{ydz-zdy}{y^2-z^2}.$$

But $dy = \dfrac{dx}{2\sqrt{1+x}}$, $dz = -\dfrac{dx}{2\sqrt{1-x}}$. Making these sub-

stitutions, and observing that $y^2 - z^2 = 2x$, the result is

$$du = -\frac{dx}{x\sqrt{1-x^2}}.$$

Ex. 15. $u = l'\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}$. Let $z = \sqrt{x^2+1}$,

$$\therefore dz = \frac{xdx}{\sqrt{x^2+1}},$$

$$u = l'(z-1) - l'(z+1),$$

$$\therefore du = \frac{dz}{z-1} - \frac{dz}{z+1},$$

$$\therefore du = \frac{2dx}{x\sqrt{x^2+1}}.$$

Ex. 16. If $u = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$, $de^{x\sqrt{-1}} = e^{x\sqrt{-1}}dx \cdot \sqrt{-1}$,

and $de^{-x\sqrt{-1}} = -e^{-x\sqrt{-1}}dx\sqrt{-1}$.

Hence $du = \left(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right)\sqrt{-1} \cdot dx$,
e is supposed here to be the hyperbolic base (21.).

Ex. 17. If $u = \cos.mx$, $\therefore du = -\sin.mx.d(mx)$ (28.);
hence $\frac{du}{dx} = -m \sin. mx$.

Ex. 18. If $u = \sin.(x^m)$, $\therefore du = \cos.(x^m) \cdot d(x^m)$ (27.);
but $d(x^m) = mx^{m-1}dx$. Hence

$$\frac{du}{dx} = mx^{m-1}\cos.(x^m).$$

Ex. 19. If $u = \sin.(a+x)$, $du = \cos.(a+x).d(a+x)$;
but $d(a+x) = dx$, $\therefore \frac{du}{dx} = \cos.(a+x)$.

Ex. 20. If $u = \cos. x + \sqrt{-1}\sin. x$, $\therefore$

$$\frac{du}{dx} = -\sin x + \sqrt{-1}\cdot\cos. x,$$

$$\text{or } \frac{du}{dx} = \sqrt{-1}\cdot\{\cos. x + \sqrt{-1}\sin. x\}.$$

Hence in this case $du = \sqrt{-1} \cdot udx$. It appears from this and Ex. 16, that the differentials of the function $e^{x\sqrt{-1}}$, and the above are the same. It will appear by the integral calculus, that these functions are actually equal.

Ex. 21. If $u = \sin.x \cos.x$, $\because$ $du = \cos.x.d \sin. x +$ $\sin.x.d \cos. x$, which, by substituting for $d \sin. x$ and $d \cos. x$, their values (27.), (28.), gives

$$\frac{du}{dx} = \cos.^2 x - \sin.^2 x = \cos. 2x.$$

Ex. 22. If $u = \sin.x \cos. a \pm \sin.a \cos.x$, $\because$

$$du = \cos.a.d \sin.x \pm \sin.a.d \cos. x,$$

$$\text{or } \frac{du}{dx} = \cos.a \cos.x \mp \sin.a \sin.x.$$

The differential and integral calculus is of very extensive use in the deduction of the formulæ one from another in Trigonometry. There are many parts, such as the expansions of series, &c. in which its application is indispensably necessary; but many even of those parts which are usually proved independently of its principles, may be much more concisely and elegantly deduced by their aid. We shall give here a few obvious examples. In the last,

$$u = \sin. (x \pm a), \because du = \cos. (x \pm a)dx, \because$$

$$\frac{du}{dx} = \cos. (x \pm a).$$

Hence $\cos. (x \pm a) = \cos.x \cos.a \mp \sin.x \sin.a.$

Ex. 23. If $u = \sin.2x$, $\because$ $du = 2 \cos.2x \cdot dx$,

$$\because \frac{du}{dx} = 2 \cos. 2x.$$

By this and Ex. 21, it follows, that if $\sin. 2x = 2 \sin.x \cos.x$,

$$\cos.2x = \cos.^2 x - \sin.^2 x.$$

Ex. 24. Let $u = \cos.x + \cos.2x + \cos.3x \ldots \cos.nx$. Since by Ex. 17,

$$d \cos.nx = - \sin.nx.d(nx) = - n \sin.nxdx,$$

we have

$$\frac{du}{dx} = -\{\sin.x + 2\sin.2x + 3\sin.3x \; . \; . \; . \; . \; n\sin.nx\}$$

Hence the summation of the first series necessarily determines that of the second.

Ex. 25. Let

$$u = \sin.\, x + \sin.\, 2x + \sin.\, 3x \ldots . + \sin.\, nx,$$

$$\therefore \frac{du}{dx} = \cos.x + 2\cos.2x + 3\cos.3x \ldots . + n\cos.nx.$$

The same remark applies as in the last example.

Ex. 26. By differentiating

$$2\cos.mx = (2\cos.x)^m - m(2\cos.x)^{m-2} + \frac{m\cdot m-3}{1\cdot 2}(2\cos.\, x)^{m-4}$$

$$\frac{m.m-4.m-5}{1\cdot 2\cdot 3}(2\cos.\, x)^{m-6}, \text{ \&c.}$$

the result, after dividing both by $2m$ and changing the signs, is

$$\sin.mx = \sin.x\,\{(2\cos.x)^{m-1} - (m-2)\,(2\cos.x)^{m-3}$$

$$+\frac{m-3\cdot m-4}{1\cdot 2}(2\cos.\, x)^{m-5} \ldots .\}$$

In general, when the summation of any trigonometrical series, or the expansion of any trigonometrical formula, is known, other series may be derived by differentiation.

Ex. 27. If $u = (l'x)^n$. Let $z = l'x$, $\therefore \dfrac{dz}{dx} = \dfrac{1}{x}$.

Also $u = z^n$, $\therefore \dfrac{du}{dz} = nz^{n-1}$, $\therefore \dfrac{du}{dx} = \dfrac{n(l'x)^{n-1}}{x}$.

Ex. 82. Let $u = l'^2x$. (This notation signifies the logarithm of $l'x$. In like manner the logarithm of l'^2x is l'^3x, and so on). Let $z = l'x$, $\therefore \dfrac{dz}{dx} = \dfrac{1}{x}$. Also $u = l'z$, $\therefore$ $\dfrac{du}{dz} = \dfrac{1}{z}$. Hence (16),

$$\frac{du}{dx} = \frac{1}{zx} = \frac{1}{xl'x}.$$

Ex. 29. If $u = l^n x$. Let $y' = l'x$, $y'' = l'y' = l^2 x$, $y''' = l'y'' = l^3 y$, and so on. Hence

$$\frac{dy'}{dx} = \frac{1}{x},$$

$$\frac{dy''}{dy'} = \frac{1}{y'} = \frac{1}{l'x},$$

$$\frac{dy'''}{dy''} = \frac{1}{y''} = \frac{1}{l^2 x},$$

. . . .

. . . .

$$\frac{dy^{(n)}}{dy^{(n-1)}} = \frac{1}{y^{(n-1)}} = \frac{1}{l^{n-1} \cdot x}.$$

Hence by the general principle in (16.),

$$\frac{du}{dx} = \frac{1}{x l'x l^2 x \ldots . l^{n-1} x}.$$

Ex. 30. If $u = z^y$. Assume the logarithms, $l'u = yl'z$,

$$\because du = u(y \frac{dz}{z} + l'zdy).$$

If in this case y and z both $= x$, the result is

$$\frac{du}{dx} = x^x(1 + l'x).$$

Ex. 31. If $u = v^{z^y}$. Let $y' = z^y$, and $\because$ $u = v^{y'}$. By the last example,

$$du = u \left\{ y' \frac{dv}{v} + l'vdy' \right\}.$$

But $dy' = y'(y \frac{dz}{z} + l'zdy)$. Hence

$$du = u \left\{ z^y \frac{dv}{v} + z^y l'v(y \frac{dz}{z} + l'zdy) \right\}.$$

If in this case v, z, and $y = x$, then $u = x^{x^x}$, and

$$\frac{du}{dx} = x^{x^x} \cdot x^x \left\{ \frac{1}{x} + l'x(1 + l'x) \right\}.$$

SECTION IV.

Of successive differentiation.

(34.) In the several functions which have been differentiated, it may be observed, that the differential coefficient is a function of the variable in general different from the primitive function. This function therefore itself may be differentiated, and another differential coefficient will be thus determined, which is called the *second differential coefficient* of the primitive function. As Lagrange calls the first differential coefficient the *first derived function*, so he calls the second differential coefficient *the second derived function*. From what has been said, it is plain that the second differential coefficient of the primitive function is the differential coefficient of the first differential coefficient considered as a function of the original variable. Let

$$\frac{du}{dx} = p,$$

$$\therefore d \cdot \frac{du}{dx} = dp.$$

Considering dx as constant, this gives

$$\frac{d(du)}{dx^2} = \frac{dp}{dx}.$$

This result is usually expressed thus,

$$\frac{d^2u}{dx^2} = \frac{dp}{dx}.$$

Thus, $\frac{d^2u}{dx^2}$ is the notation for the second differential coefficient of the primitive function u.

It should be remembered here, that d^2u does not signify $d \times d \times u$, nor does dx^2 signify the differential of x^2. The

former signifies the differential of the differential of u, and the latter the square of dx.

(35.) It has been already observed, that although the differential coefficient, being a function of x, is determinate for any proposed value of x, yet, that for any such proposed value, the differentials of u and x are indeterminate. All that is determinate in this case is the ratio $du : dx$, or the quote $\frac{du}{dx}$. This remark is of importance in successive differentiation.

(36.) Considering $\frac{du}{dx}$ as a function of x, it must be supposed to vary with x. This variation may be effected by a variation in du or dx, or in both. It contributes, however, much to the simplicity of the notation, and does not affect the generality of the results, to ascribe to du the entire variation of the function $\frac{du}{dx}$ produced by the variation of the variable x, and, consequently, to suppose dx constant. We are evidently authorised to adopt this supposition, as appears by the preceding observations: it is for this reason that in the investigation of the second differential coefficient we assume

$$d \cdot \frac{du}{dx} = \frac{d^2u}{dx},$$

the factor $\frac{1}{dx}$ being constant (18.). The variable, whose differential is considered constant, is called *the independent variable.*

(37.) The second differential coefficient, like the first, being in general a function of the original variable, is susceptible of differentiation, from whence results a *third differential coefficient,* or according to Lagrange, a *third derived function.* Since dx has been considered constant in

determining the second differential coefficient, it must continue to be so considered in deriving the third differential coefficient from the second, ∵

$$d\frac{d^2u}{dx^2}=\frac{d(d^2u)}{dx^2}.$$

The notation for $d(d^2u)$ is d^3u. Let $d\cdot\frac{d^2u}{dx^2}=qdx$, hence

$$\frac{d^3u}{dx^3}=q,$$

which is therefore the third differential coefficient.

In like manner the fourth, fifth, &c. differential coefficients may be determined, the general notation for the nth differential coefficient being $\frac{d^nu}{dx^n}$.

PROP. XVII.

(38.) *Three quantities being so related that the first* u *is a function of the second* y, *and the second* y *is a function of the third* x, *given the second differential coefficients of the first as a function of the second, and of the second as a function of the third, to determine the second differential coefficient of the first as a function of the third.*

In this case $\frac{d^2u}{dy^2}$, and $\frac{d^2y}{dx^2}$ are given, to find $\frac{d^2u}{dx^2}$. The coefficients $\frac{d^2u}{dy^2}$ and $\frac{d^2y}{dx^2}$ in their present state imply a contradiction, for the first depends on the supposition that dy is constant, and the second owes its existence to the variation of dy. To reconcile this, it will be necessary to substitute for $\frac{d^2u}{dy^2}$ what it would have been if dy had not been considered constant. For this purpose it should be remembered that $\frac{d^2u}{dy^2}$ was derived from the operations indicated by

$$\frac{d \cdot \frac{du}{dy}}{dy},$$

having been performed, supposing dy constant. Now let dy be supposed variable, and the formula becomes

$$\frac{dyd^2u - dud^2y}{dy^3},$$

which is therefore the second differential coefficient when y is not supposed to be the independent variable. Substituting in this for dy and d^2y their values derived from considering y a function of x, the result will be the second differential coefficient of u as a function of x.

It appears, therefore, that where several variables are each a function of the other, only one of them ought to be considered as an independent variable in differentiation. This, however, need not be attended to unless the differential coefficients of two or more of the functions related to *different* independent variables enter the same formula. In that case, all the independent variables but one must be removed by the method given above, which may easily be extended to differential coefficients of superior orders.

PROP. XVIII.

(39.) *To determine the successive differential coefficients of a power.*

Let $u = x^m$. By (24),

$$\frac{du}{dx} = mx^{m-1},$$

$$\therefore \frac{d^2u}{dx^2} = m \cdot m - 1 \cdot x^{m-2},$$

$$\frac{d^3u}{dx^3} = m \cdot m - 1 \cdot m - 2 \cdot x^{m-3}.$$

. . . .

And, in general, the nth differential coefficient is

$$\frac{d^n u}{dx^n} = m \cdot m-1 \cdot m-2 \ldots \overline{m-(n-1)} \cdot x^{m-n}.$$

The differential coefficient of the mth order when m is a positive integer, is

$$\frac{d^m u}{dx^m} = m \cdot m-1 \cdot m-2 \ldots 3 \cdot 2 \cdot 1.$$

This being a constant quantity, all succeeding differential coefficients are $= 0$. But if m be either negative or fractional, the factor $m - (n-1)$ can never $= 0$, and therefore the differential coefficients never $= 0$.

PROP. XIX.

(40.) *To determine the successive differentials of the product of two functions.*

Let $u = yy'$, $\because du = ydy' + y'dy$, $\because$

$$d^2u = yd^2y' + 2dydy' + y'd^2y,$$
$$d^3u = yd^3y' + 3dyd^2y' + 3dy'd^2y + y'd^3y,$$
$$d^4u = yd^4y' + 4dyd^3y' + 6d^2yd^2y' + 4d^3ydy' + y'd^4y.$$

. . . .

The law of the exponents and coefficients is obviously that of the binomial series; therefore, in general,

$$d^n u = yd^n y' + ndyd^{n-1}y' + \frac{n \cdot n-1}{1 \cdot 2} d^2y\, d^{n-2}y'$$
$$+ \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} d^3yd^{n-3}y' + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} d^4yd^{n-4}y' \text{ \&c.}$$

As an example of the application of this theorem, let it be required to find the fourth differential of $z^2 - x^2$. Let

$y = z + x$, and $y' = z - x$, $\because d^n y = d^n z + d^n x$, and

$d^n y' = d^n z - d^n x$, $\because$

$$
\begin{aligned}
d^4(z^2-x^2) = {} & (z+x)(d^4z-d^4x)+4(dz+dx)(d^3z-d^3x) \\
& +6(d^2z+d^2x)(d^2z-d^2x)+4(d^3z+d^3x)(dz-dx) \\
& +(z-x)(d^4z+d^4x) = 2\{zd^4z+4dzd^3z+3d^2zd^2z-xd^4x \\
& -4dxd^3x-3d^2xd^2x\}.
\end{aligned}
$$

PROP. XX.

(41.) *To determine the successive differential coefficients of an exponential function.*

Let $u = a^x$. By (20.) $\frac{du}{dx} = ku$; $\frac{1}{k}$ being the modulus to the base a. Hence

$$\frac{d^2u}{dx} = kdu.$$

And by substituting for du its value $kudx$, and dividing by dx,

$$\frac{d^2u}{dx^2} = k^2u.$$

In like manner

$$\frac{d^3u}{dx^3} = k^3u,$$

and in general,

$$\frac{d^nu}{dx^n} = k^nu.$$

If a be the hyperbolic base $k = 1$; and in this case the differential coefficients are all equal to the primitive function u.

PROP. XXI.

(42.) *To determine the successive differential coefficients of a logarithm.*

Let $u = lx$, $\because \frac{du}{dx} = \frac{1}{x} = x^{-1}$. Hence

$$\frac{d^2u}{dx^2} = -\frac{1}{x^2} = -x^{-2},$$

$$\frac{d^3u}{dx^3} = 2x^{-3},$$

$$\frac{d^4u}{dx^4} = -\ 2.3.x^{-4}.$$

. . . .

. . . .

. . . .

The differential coefficients are therefore alternately positive and negative, and that of the nth order is,

$$1 \cdot 2 \cdot 3 \ldots \overline{n-1} \cdot x^{-n},$$

which is + if n be odd, and − if n be even.

In this case the logarithm is assumed to be hyperbolic. If it be not, the successive differential coefficients should be affected by the modulus as a factor.

PROP. XXII.

(43.) *To determine the successive differential coefficients of the sine and cosine as functions of the arc.*

Let $u = \sin.x$, ∵

$$\frac{du}{dx} = \cos.x,$$

$$\because \frac{d^2u}{dx^2} = -\sin.x,$$

$$\frac{d^3u}{dx^3} = -\cos.x.$$

. . . .

. . . .

And in general, if n be an odd number,

$$\frac{d^nu}{dx^n} = \pm \cos.x;$$

+ being taken when $\frac{n-1}{2}$ is even, and −, when $\frac{n-1}{2}$ is odd. And when n is even,

$$\frac{d^n u}{dx^n} = \pm \sin.x.$$

$+$ being taken when $\frac{n}{2}$ is even, and $-$, when $\frac{n}{2}$ is odd.

Since $\cos. x = \sin. (\frac{\pi}{2} - x)$, and $-dx = d(\frac{\pi}{2} - x)$ it follows that by changing the sine into the cosine, and $+$ into $-$ and *vice versa*, the preceding observations may be applied to the successive differential coefficients of the cosine. Hence if $u = \cos. x$,

$$\frac{d^n u}{dx^n} = \pm \sin.x;$$

when n is odd, $+$ being used when $\frac{n-1}{2}$ is odd, and $-$ when $\frac{n-1}{2}$ is even. And

$$\frac{d^n u}{dx^n} = \mp \cos.x.$$

When n is even, $+$ being used when $\frac{n}{2}$ is even, and $-$, when $\frac{n}{2}$ is odd.

PROP. XXIII.

(44.) *To determine the successive differential coefficients of the tangent and cotangent as functions of the arc.*

Let $u = \tan.x$, $\because \frac{du}{dx} = \frac{1}{\cos.^2 x} = \sec.^2 x$; hence

$$\frac{d^2u}{dx^2} = 2 \sec.x d \sec.x = 2 \sec.^2 x \tan.x dx,$$

$$\because \frac{d^2u}{dx^2} = 2 \tan.x\,(1 + \tan.^2 x),$$

$$\text{or } \frac{d^2u}{dx^2} = 2u(1 + u^2).$$

Hence the third differential is

$$\frac{d^3u}{dx^2} = 2(1 + 3u^2)du.$$

But $du = (1 + u^2)dx$, $\therefore$

$$\frac{d^3u}{dx^3} = 2(1 + u^2)\,(1 + 3u^2).$$

By continuing this process, the succeeding coefficients may in like manner be found.

The differential coefficients of the cot. x may be deduced from those of the tangent by changing x into $(\frac{\pi}{2} - x)$, and changing the sign of dx.

PROP. XXIV.

(45.) *To find the successive differentials of the secant and cosecant as functions of the arc.*

Let $u = \sec.x$, $\therefore$ $du = \tan.x \sec.x dx$, $\therefore$

$$\frac{du}{dx} = u\,\sqrt{u^2 - 1},$$

$$\therefore \frac{d^2u}{dx} = \sqrt{u^2 - 1}\cdot du + \frac{u^2du}{\sqrt{u^2-1}}.$$

Substituting for du its value already found, we have

$$\frac{d^2u}{dx^2} = u(2u^2 - 1) = 2u^3 - u.$$

Differentiating again, we find

$$\frac{d^3u}{dx^2} = 2\cdot3\cdot u^2du - du,$$

$$\text{or } \frac{d^3u}{dx^3} = 2\cdot3\cdot u^3\sqrt{u^2-1} - u\sqrt{u^2-1} = u\sqrt{u^2-1}\cdot(2\cdot3u^2-1),$$

and in a similar way the process may be continued.

To find the coefficients of $u = \text{cosec.}\, x$, it is only necessary to change x into $(\frac{\pi}{2} - x)$, and change the sign of dx in the former results.

PROP. XXV.

(46.) *To determine the successive differential coefficients of the arc as a function of its sine and cosine.*

Let $u = \sin.^{-1} x$, $\therefore \frac{du}{dx} = (1 - x^2)^{-\frac{1}{2}}$. By differentiating this successively, we find

$$\frac{d^2u}{dx^2} = x(1 - x^2)^{-\frac{3}{2}},$$

$$\frac{d^3u}{dx^3} = (1 - x^2)^{-\frac{3}{2}} + 3x^2(1 - x^2)^{-\frac{5}{2}},$$

$$\frac{d^4u}{dx^4} = 3^2x(1 - x^2)^{-\frac{5}{2}} + 3 \cdot 5x^3(1 - x^2)^{-\frac{7}{2}},$$

$$\frac{d^5u}{dx^5} = 3^2(1 - x^2)^{-\frac{5}{2}} + 2 \cdot 5 \cdot 9x^2(1 - x^2)^{-\frac{7}{2}} + 3 \cdot 5 \cdot 7x^4(1 - x^2)^{-\frac{9}{2}}.$$

. . . .

And so the process may be continued.

If $u = \cos.^{-1}x$, the successive differential coefficients have the same form but different signs.

PROP. XXVI.

(47.) *To determine the successive differential coefficients of an arc as a function of its tangent or cotangent.*

Let $u = \tan.^{-1}x$, $\therefore \frac{du}{dx} = (1 + x^2)^{-1}$. Hence

$$\frac{d^2u}{dx^2} = -2x(1 + x^2)^{-2},$$

$$\frac{d^3u}{dx^3} = -\,2(1+x^2)^{-2} + 2\,.\,4\,.\,x^2(1+x^2)^{-3},$$

$$\frac{d^4u}{dx^4} = 2^3.3.x(1+x^2)^{-3} - 2^4.3.x^3(1+x^2)^{-4},$$

$$\frac{d^5u}{dx^5} = 2^3.3(1+x^2)^{-3} - 2^5.3^2x^2(1+x^2)^{-4} + 2^7.3.x^4(1+x^2)^{-5},$$

. . . .

And in this manner the process may be continued.

The coefficients for $u = \cot.^{-1}x$ may be found by substituting $(\frac{\pi}{2} - x)$ for x, and $-\,dx$ for $+\,dx$.

PROP. XXVII.

(48.) *To determine the successive differential coefficients of an arc considered as a function of its secant or cosecant.*

Let $u = \sec.^{-1}x$, $\because du = \dfrac{dx}{x\sqrt{x^2-1}}$, $\because$

$$\frac{du}{dx} = x^{-1}(x^2-1)^{-\frac{1}{2}},$$

$$\frac{d^2u}{dx^2} = -\,x^{-2}(x^2-1)^{-\frac{1}{2}} - (x^2-1)^{-\frac{3}{2}},$$

$$\frac{d^3u}{dx^3} = +\,2x^{-3}(x^2-1)^{-\frac{1}{2}} + x^{-1}(x^2-1)^{-\frac{3}{2}} +$$

$$3x(x^2-1)^{-\frac{5}{2}}.$$

. . . .

and so on. The coefficients of $u = \text{cosec.}^{-1}x$ may be found from these as in the former cases.

SECTION V.

Of development. The theorems of Taylor, Maclaurin, Lagrange, and Laplace.

(49.) One of the most important uses of the calculus is in furnishing theorems by which a function may be reduced to a series of monomes, of which the powers of any proposed quantity which enters the function shall be factors, the other factors of each monome being independent of this quantity. All the different theorems named at the head of this section have this object. We shall therefore proceed to investigate them in the order in which they have been stated above.

PROP. XXVIII.

(50.) *If the variable of a function be supposed to consist of two parts,* y *and* h, *the differential coefficient will be the same to whichever part the variation be ascribed.*

Let $u = \text{F}(x)$, and let $x = y + h$, $\because u = \text{F}(y + h)$. Let $\frac{du}{dx} = \text{F}'(x)$, $\because \frac{du}{dx} = \text{F}'(y + h)$. Now if the variation of x be ascribed entirely to h, and y be considered constant, $dx = d(y + h) = dh$, $\because \frac{du}{dh} = \text{F}'(y + h)$. If, on the other hand, y be considered variable and h constant,

$$dx = d(y + h) = dy, \because \frac{du}{dy} = \text{F}'(y + h), \because \frac{du}{dy} = \frac{du}{dh}:$$

and the same reasoning may be applied to the successive differential coefficients.

PROP. XXIX.

TAYLOR'S THEOREM.

(51.) *The variable of a function being supposed to consist of two parts* x *and* h, *to develop the function in a series of powers of one of the parts* h.

Let the function be $F(x+h)$, and let its successive differential coefficients determined by considering x as variable and h constant be $F^1(x+h)$, $F^2(x+h)$, $F^3(x+h)$ . . . .

Let the proposed development be

$$F(x+h) = Ah^a + Bh^b + Ch^c + \dots$$

the exponents being arranged in ascending order.

Let this be differentiated considering h as variable, and by the notation explained in (37),

$$\frac{dF(x+h)}{dh} = F^1(x+h),$$

$$\frac{d^2F(x+h)}{dh^2} = F^2(x+h),$$

. . . .

. . . .

Hence

$$F(x+h) = Ah^a + Bh^b + Ch^c + Dh^d \dots\dots\dots\dots [1],$$

$$F^1(x+h) = aAh^{a-1} + bBh^{b-1} + cCh^{c-1} + dDh^{d-1} \dots\dots [2],$$

$$F^2(x+h) = (a(a-1)Ah^{a-2}) + (b(b-1)B^{b-2}) + (c(c-1)Ch^{c-2}) + \dots [3],$$

$$F^3(x+h) = (a(a-1)(a-2)Ah^{a-3}) + (b(b-1)(b-2)Bh^{b-3}) + (c(c-1)(c-2)Ch^{c-3}) \dots [4],$$

. . . .

. . . .

When $h = 0$, the functions on the left of these equations become $F(x)$ and its successive differential coefficients, $F^1(x)$, $F^2(x)$, $F^3(x)$ . . . .

In order to determine the coefficients and exponents of the series [1], it will be necessary to consider,

1°. The case where the value assigned to x in $F(x+h)$ is not a root of any of the equations

$$F(x) = 0,\ F^1(x) = 0,\ F^2(x) = 0 \ldots\ldots \quad [5],$$
$$f(x) = 0, f^1(x) = 0, f^2(x) = 0 \ldots\ldots \quad [6].$$

Where $f(x), f^1(x), f^2(x) \ldots\ldots$ denote the reciprocals of $F(x)$, $F^1(x)$, $F^2(x) \ldots\ldots$ In other words, we shall in this case suppose some value assigned to x, which does not render the function $F(x)$ or any of its differential coefficients either nothing or infinite.

2°. The case where the value assigned to x is a root of one or more of the equations

$$F(x) = 0,\ F^1(x) = 0,\ F^2(x) = 0 \ldots\ldots$$

3°. The case where the value assigned to x is a root of one or more of the equations

$$f(x) = 0, f^1(x) = 0, f^2(x) = 0 \ldots\ldots$$

4°. The case where the value assigned to x is a root of several of each of these systems of equations.

(52.) 1°. In this case $a = 0$. For if $a > 0$, $Ah^a = 0$ when $h = 0$, and since the exponents $a, b, c, \ldots\ldots$ are ascending if the first be > 0, they must all be > 0, $\therefore h = 0$ renders every term of the series $= 0 \therefore F(x) = 0$. Such a value of x being excluded, a cannot in this case be > 0.

If $a < 0$, Ah^a would be infinite when $h = 0$, $\therefore F(x)$ would be infinite. But such a value being excluded from this case, a cannot be < 0. Since therefore a cannot be > 0 nor < 0, $\therefore a = 0$.

All the succeeding exponents being > 0, $h = 0$ renders all the succeeding terms $= 0 \therefore A = F(x)$. Thus the first coefficient and exponent is determined.

The series [2] becomes therefore

$$F^1(x+h) = bBh^{b-1} + cCh^{c-1} + dDh^{d-1} \ldots\ldots$$

If $h = 0$, $F^1(x+h)$ becomes $F^1(x)$, and since by supposition no value is assigned to x which renders $F^1(x) = 0$, $b - 1$ cannot be > 0; and since no value is assigned to x

which renders $F^1(x)$ infinite, $b-1$ cannot be <0. These follow in the same manner as for the first exponent a. Hence $b-1=0 \therefore b=1$. Since the exponents ascend $c-1, d-1, \ldots$ are >0, $\therefore h=0$ gives $B=F^1(x)$. Thus the second coefficient and exponent are determined.

The series [3] therefore becomes

$$F^2(x+h)=c\,.\,c-1\,.\,Ch^{c-2}+d\,.\,d-1\,.\,Dh^{d-2}\ldots$$

If $h=0$, $F^2(x+h)$ becomes $F^2(x)$, and since no value is assigned to x which renders this either infinite or nothing; it follows as before, that $c-2$ is neither >0 nor <0, $\therefore c-2=0 \therefore c=2$. Since $d-2, e-2 \ldots$ are >0, $\therefore h=0$ gives $2C=F^2(x) \therefore C=F^2(x)\,.\,\frac{1}{2}$. Thus the third coefficient and exponent are determined.

The series [4] becomes

$$F^3(x+h)=d.d-1.d-2.Dh^{d-3}+e.e-1.e-2.Eh^{e-3}\ldots$$

If $h=0$, $F^3(x+h)$ becomes $F^3(x)$, and it follows in a similar way that $d-3$ can neither be >0 nor <0, $\therefore d=3$. And also $D=F^3(x)\,.\,\frac{1}{1.2.3}$. Thus the fourth exponent and coefficient are determined.

In the same way the others may be found, and the several values being substituted in [1] for the coefficients and exponents, the series becomes

$$F(x+h)=F(x)+F^1(x).\frac{h}{1}+F^2(x).\frac{h^2}{1.2}+F^3(x).\frac{h^3}{1.2.3}+ F^4(x)\frac{h^4}{1.2.3.4} \ldots \ldots \quad [7].$$

Or if $u=F(x)$ and $u'=F(x+h)$ the series may be expressed

$$u'=u+\frac{du}{dx}\cdot\frac{h}{1}+\frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3}+\frac{d^4u}{dx^4}\cdot\frac{h^4}{1.2.3.4}\ldots \quad [8].$$

(53.) If $\Delta u=u'-u$, and $h=\Delta x$, and the arbitrary quan-

tity dx be supposed to equal Δx, the quantities du, d^2u, d^3u, &c. consequently, having such values as will render $\frac{du}{\Delta x}$, $\frac{d^2u}{\Delta x^2}$, $\frac{d^3u}{\Delta x^3}$, &c. equal to the successive differential coefficients, we have

$$\Delta u = \frac{du}{1} + \frac{d^2u}{1.2} + \frac{d^3u}{1.2.3} + \frac{d^4u}{1.2.3.4} + \&c. \ldots$$

which expresses the *difference* of the function in a series of its *corresponding* successive *differentials*. The character Δ before a variable signifies its finite difference.

The series which is the result of this investigation was first published by Dr. Brook Taylor in his *Methodus Incrementorum* in 1715. Taylor was a profound mathematician of the old school; he does not, however, seem at all aware of the immense importance of his own discovery. Lagrange has made it the basis of his theory of analytic functions. On it depend almost the entire application of the calculus to geometry, the principles of contact, osculation, singular points, &c. &c. Some very elegant applications of it have been made by another able modern mathematician in finding fluxions *per saltum*, in approximating to the roots of equations, &c. *

(54.) II. If the value of x be a root of the equation $F(x) = 0$, the development [7] wants its first term, but otherwise remains unchanged. If x be a root of $F^1(x) = 0$, the development wants the second term, and in general if x be a root of $F^n(x) = 0$, the series wants the $(n + 1)$th term. If x be a common root of several of the equations [5], the series will want the corresponding terms. It appears therefore that these particular values of x do not form *exceptions* to the development [7].

(55.) III. If the value of x be a root of the equation

* See Dr. Brinkley's Essay, Tran. Royal Irish Academy, vol. 7.

$f(x) = 0$, the development [7] becomes inapplicable, because all its coefficients become infinite, and $F(x + h)$ is expressed by a series of infinite monomes. It is easy to perceive that any value of x which is a root of $f(x) = 0$ must also be a common root of $f^1(x) = 0, f^2(x) = 0 \ldots$ For if x render $F(x)$ infinite, the exponent a in [1] must be negative, $\because$ the exponent $a - 1$ in [2], $a - 2$ in [3], $a - 3$ in [4], &c. must be also negative. Hence $h = 0$ must render all these infinite; but these become in this case $F^1(x)$, $F^2(x)$, $F^3(x)$, &c. $\because$ &c.

The values of the coefficients and exponents of the series must be in this case determined by the common algebraical methods. They may also be determined in the following manner. The exponent a being negative, the series is

$$F(x + h) = Ah^{-a} + Bh^b + Ch^c \ldots$$

the succeeding exponents $b, c, \ldots$ being either positive quantities or negative quantities $< a$.

To determine the value of a, let such a power of h be found, which being multiplied into the given function $F(x + h)$ will give a product which becomes neither $= 0$, nor infinite when $h = 0$, the exponent of this power will be a. For if not, let it be k, $\because$

$$F(x + h) \cdot h^k = Ah^{k-a} + Bh^{k+b} + Ch^{k+c} \ldots$$

It is evident that the exponents of this series ascend, and if $k - a$ is positive, they are all positive, $\because$ all the terms vanish when $h = 0$, $\because$

$$F(x + h)h^k = 0;$$

when $h = 0$, which is contrary to hypothesis.

If k were less than a, $k - a$ would be negative, and therefore $F(x + h)h^k$ would be infinite when $h = 0$, which is also contrary to hypothesis. Hence $k = a$. The exponent a being thus found, the coefficient A is what $F(x + h) \cdot h^{-a}$ becomes when $h = 0$.

Having thus determined A and a, the first term of the

development becomes known. Let it be brought over, so that

$$\text{F}(x+h) - \text{A}h^{-a} = \text{B}h^{b} + \text{C}h^{c} + \text{D}h^{d} \ldots.$$

The quantity on the left of this equation being known; if b be negative, it may be found by the same process as that used to determine a. It may be known whether it be negative by determining if $h = 0$ render $\text{F}(x+h) - \text{A}h^{-a}$ infinite. If $h = 0$ render this $= 0$, then $b > 0$, and we shall presently explain the method of determining it. If $h = 0$ do not render $\text{F}(x+h) - \text{A}h^{-a}$ either $= 0$ or infinite, then $b = 0$, and the value of $\text{F}(x+h) - \text{A}h^{-a}$ when $h = 0$ is the value of B. The other exponents, when negative, and coefficients may be determined in a similar way.

If x be a root of $f^{1}(x) = 0$, but not of $f(x) = 0$, then the series [1] and [2] become

$$\text{F}(x+h) = \text{F}(x) + \text{B}h^{b} + \text{C}h^{c} + \text{D}h^{d} \ldots.$$
$$\text{F}^{1}(x+h) = b\text{B}h^{b-1} + c\text{C}h^{c-1} + d\text{D}h^{d-1} \ldots.$$

Since $\text{F}^{1}(x+h)$ is infinite when $h = 0$, $b - 1$ must be < 0. But since x is not a root of $f(x)$, b must be > 0, $\therefore b$ must be a proper fraction and positive. To determine its value, let $\text{F}(x)$ be brought over in the first, $\therefore$

$$\text{F}(x+h) - \text{F}(x) = \text{B}h^{b} + \text{C}h^{c} + \text{D}h^{d} \ldots.$$

let that power of h be found, by which $\text{F}(x+h) - \text{F}(x)$ being divided, the quote will neither vanish nor become infinite when $h = 0$. The exponent of that power will be $= b$. For let it be k,

$$\frac{\text{F}(x+h) - \text{F}(x)}{h^{k}} = \text{B}h^{b-k} + \text{C}h^{c-k} + \ldots.$$

If $k < b$, $h = 0$ renders this $= 0$, which is contrary to hypothesis, and if $k > b$, $h = 0$ renders it infinite, which is also contrary to hypothesis, $\therefore k = b$.

If x be a root of $f^{2}(x) = 0$, but not of $f(x) = 0$, nor $f^{1}(x) = 0$, then the series [1], [2], [3], become

$$\text{F}(x+h) = \text{F}(x) + \text{F}^1(x)\frac{h}{1} + \text{C}h^c + \text{D}h^d \ldots.$$

$$\text{F}^1(x+h) = \text{F}^1(x) + c\text{C}h^{c-1} + d\text{D}h^{d-1} \ldots.$$

$$\text{F}^2(x+h) = c\,.\,c-1\,.\,\text{C}h^{c-2} + d\,.\,d-1\,.\,\text{D}^{d-2}.\ldots.$$

Since $\text{F}^2(x+h)$ becomes infinite when $h=0$, $\because c-2<0$, $\because c<2$.

But since the exponents ascend, $c>1$. Hence the value of c is between 1 and 2. It may be thus determined in the same manner as b.

$$\frac{\text{F}(x+h)-\text{F}(x)-\text{F}^1(x).\frac{h}{1}}{h} = \text{C}h^{c-1} + \text{D}h^{d-1} + \ldots.$$

The left side of this equation is known. Let that fractional power of h be found, by which this being divided, gives a quote which neither vanishes nor becomes infinite when $h=0$. The exponent of this power is $c-1$. Hence c becomes known, and also C.

It follows therefore in general, that if a value be assigned to x which is a root of $f^n(x)=0$, or which renders the nth differential coefficient infinite, but none of the preceding ones, the series [7] gives the true development as far as the nth term inclusive; but the exponent of h in the $(n+1)$th term is a fraction, whose value is between the integers n and $n+1$, and which may be determined by the method already explained*.

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* The method of determining the exponents of h given above is taken from the *Theorie des Fonctions Analytiques* of Lagrange. This method applied to negative exponents may be somewhat improved by the application of the Integral Calculus. Let a be negative, and the series is

$$\text{F}(x+h) = \text{A}h^{-a} + \text{B}h^b + \text{C}h^c + \ldots.$$

Multiply both by dh, and integrate

$$\int \text{F}(x+h)dh = \frac{1}{1-a}.\text{A}h^{1-a} + \frac{1}{1+b}.\text{B}h^{1+b} + \frac{1}{1+c}\text{C}h^{1+c}.\ldots.$$

(56.) There are some peculiar circumstances attendant upon the state of the function when x receives any value which is to be found among the roots of the equations

$$f(x) = 0, f^1(x) = 0, f^2(x) = 0 \ldots .$$

which merit examination.

If the denominator of any fractional exponents which occur in the development of $F(x+h)$ be an even integer, the numerator (the fraction being supposed in its least terms) must be odd. The power of h therefore being the even root of an odd power is imaginary if h be negative, and has two real values with different signs if h be affirmative. Hence the particular state of the function $F(x+h)$ is one at which it passes from a real to an impossible value or *vice versa* by the variation of x. In this transition it is plain that two values become equal and then impossible, which must happen by a radical disappearing in the value of the function corresponding to the particular value of x, which renders the differential coefficient infinite. This circumstance is similar to that which will be shown to happen (114), when some differential coefficient assumes the form $\frac{0}{0}$. But there is a very important distinction to be observed between the cases. In the one case the radical

---

Let $\int F(x+h)dh = F_{\prime}(x+h)$. Multiply again by dh, and integrating

$$\int F_{\prime}(x+h)dh = \frac{1}{1-a} \cdot \frac{1}{2-a} \cdot Ah^{2-a} + \frac{1}{1+b} \cdot \frac{1}{2+b} \cdot Bh^{2+b} \ldots\ldots$$

Let this process be continued until an integral be found, which will neither vanish or be finite when $h = 0$. If one be found which vanishes when $h = 0$, a is a fraction whose value is between the number of integrations and the integer next below it. If it be finite, then a is equal to the number of integrations. This is evident.

passes through zero without becoming impossible on either side of it, therefore it must vanish in the primitive function, not by its suffix vanishing, for that would infer a change of sign in the suffix, and therefore a transition from a real to imaginary value, but by a coefficient of the radical vanishing which produces a change of sign in the term in which the radical was engaged without rendering the radical imaginary. In the present case, however, the function passes from a real to an imaginary state, and therefore the particular value of x must make the suffix of the radical vanish, and not a coefficient of it, and the suffix changing its sign in passing through zero, there is a transition of the function from a real to an imaginary state, or *vice versa*.

If the denominator of the lowest fractional power which occurs in the development be an odd number, the numerator may be either even or odd. First suppose it even. The sign of the fractional power of h in this case is the same whether h be positive or negative, and therefore this term of the development of $\mathrm{F}(x+h)$ and $\mathrm{F}(x-h)$ is the same; and in each case there is but one real value for the power of h. If the numerator be an odd number, the sign of the power of h changes with the sign of h, and therefore for $\mathrm{F}(x+h)$, and $\mathrm{F}(x-h)$, the fractional power of h has different values, but in each case has but one real value.

PROP. XXX.

MACLAURIN'S THEOREM.

(57.) *To expand a function in a series of ascending integral and positive powers of the variable.*

Let $u = \mathrm{F}(x)$ and $u' = \mathrm{F}(x+h)$. If $x = 0$ in the equation [8], it becomes

$$u' = A_0 + A_1 \cdot \frac{h}{1} + A_2 \cdot \frac{h^2}{1.2} + A_3 \frac{h^3}{1.2.3} + A_4 \frac{h^4}{1.2.3.4} \cdots$$

where A_0, A_1, A_2, A_3, &c. are what u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$, &c. become when $x = 0$. When $x = 0$, $u = F(h)$, and therefore the differential coefficients of this function must be the same functions of h as those of $F(x)$ are of x. Hence it follows that the latter when $x = 0$ become identical with the former when $h = 0$. The quantities A_0, A_1, A_2, &c. are what the function $F(h)$ and its differential coefficients become when $h = 0$. Hence the above series, considering $u = F(h)$, solves the problem. In general, therefore,

$$F(x) = A_0 + A_1 \cdot \frac{x}{1} + A_2 \cdot \frac{x^2}{1.2} + A_3 \cdot \frac{x^3}{1.2.3} + \cdots$$

where A_0, A_1, A_2, A_3, ... are what the function $F(x)$ and its differential coefficients become when $x = 0$.

This theorem, like that of Taylor, is liable to exceptions; but the exceptions arise here from the form of the function, and not, as in the former case, from the particular value assigned to the variable. Taylor's series, if x be indeterminate, holds good *without exception;* but that of Maclaurin, even though x be considered indeterminate, is liable to exceptions, because the coefficients are not functions of x, but are what certain functions of x become when $x = 0$, in which case they may happen to be infinite or impossible. Thus, if the function to be expanded be $\frac{1}{x}$, the first term A_0 being $\frac{1}{0}$ is infinite, and the function cannot be expanded in the required form. This, however, ought not to be called a *fault* or *failure* in the theorem, because in these cases the function does not admit of an expansion in positive integral powers of the variable.

The cases which form exceptions to Maclaurin's series

may sometimes be solved by a transformation. The substitution of $x^k z$ for u, k being arbitrary, frequently effects this. Such a value should be assigned to k that none of the quantities $u\ \frac{du}{dx},\ \frac{d^2u}{dx^2}\ \ldots$ should be infinite when $x = 0$. An example of this is given in (83).

Maclaurin's theorem may likewise be applied to develop a function by descending powers of the variable. Let $u = \text{F}(x)$ be the function, and let $zx = 1$, or $x = \frac{1}{z}$. Substitute this for x, and $\because u = \text{F}\left(\frac{1}{z}\right)$, or $= f(z)$. Let this be developed by Maclaurin's theorem according to the ascending powers of z, and then substitute $\frac{1}{x}$ for z, the result will be a series of descending powers of x. For an example of this see (85).

PROP. XXXI.

LAGRANGE'S THEOREM.

(58.) *Given* $u = \text{F}(y)$ *and* $y = z + xf(y)$ *to expand* u *in a series of ascending integral and positive powers of* x, z *not being a function of* x.

Considering u as a function of x by Maclaurin's theorem,

$$u = \text{A}_0 + \text{A}_{,} \cdot \frac{x}{1} + \text{A}_2 \cdot \frac{x^2}{1.2} + \text{A}_3 \cdot \frac{x^3}{1.2.3} + \text{A}_4 \cdot \frac{x^4}{1.2.3.4} \cdots$$

Where $\text{A}_0, \text{A}_{,}, \text{A}_2, \text{A}_3$, &c. are what $u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}$, &c. become when $x = 0$. The problem will therefore be solved if the values of these be determined.

If in the equation $y = z + xf(y)$, y be considered as a function of x, we have

$$\frac{dy}{dx} = f(y) + x \cdot \frac{dy}{dx} \cdot \frac{df(y)}{dy},$$

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} \cdot \frac{df(y)}{dy} + x \cdot \frac{d^2y}{dx^2} \cdot \frac{df(y)}{dy} + x \cdot \frac{dy^2}{dx^2} \cdot \frac{d^2f(y)}{dy^2}$$

. . . .

And by the equation $u = \mathrm{F}(y)$,

$$\frac{du}{dx} = \frac{d\mathrm{F}(y)}{dy} \cdot \frac{dy}{dx},$$

$$\frac{d^2u}{dx^2} = \frac{d^2y}{dx^2} \cdot \frac{d\mathrm{F}(y)}{dy} + \frac{dy^2}{dx^2} \cdot \frac{d^2f(y)}{dy^2},$$

. . . .

If $x = 0$, the function y and its differential coefficients become

$$z, \; f(z), \; \frac{d.f(z)^2}{dz}, \; \frac{d^2.f(z)^3}{dz^2}, \dots$$

And by these substitutions, we find

$$\mathrm{A}_0 = \mathrm{F}(z),$$

$$\mathrm{A}_1 = f(z)\frac{d.\mathrm{F}(z)}{dz},$$

$$\mathrm{A}_2 = \frac{d\left\{f(z)^2 \cdot \frac{d\mathrm{F}(z)}{dz}\right\}}{dz},$$

$$\mathrm{A}_3 = \frac{d^2\left\{f(z)^3 \cdot \frac{d\mathrm{F}(z)}{dz}\right\}}{dz^2}.$$

And in general,

$$\mathrm{A}_n = \frac{d^{n-1}\left\{f(z)^n \cdot \frac{d\mathrm{F}(z)}{dz}\right\}}{dz^{n-1}}.$$

Therefore if $\frac{d\mathrm{F}(z)}{dz} = q$ and $f(z) = p$, we obtain the series,

$$u = \text{F}(z) + pq.\frac{x}{1} + \frac{d(p^2q)}{dz}.\frac{x^2}{1.2} + \frac{d^2.(p^3q)}{dz^2}.\frac{x^3}{1.2.3} + \ldots$$

or

$$u = \text{F}(z) + f(z).\frac{d.\text{F}(z)}{dz}.\frac{x}{1}.\frac{+d.\left[f(z)^2.\frac{d.\text{F}(z)}{dz}\right]}{dz}.\frac{x^2}{1.2} +$$

. . . .

which is the solution of the proposed problem.

(59.) *Cor.* 1. If $f(y) = 1$, and $\because f(z) = 1$, and $x = h$, this series becomes

$$u = \text{F}(z+h) = \text{F}(z) + \frac{d.\text{F}(z)}{dz}.\frac{h}{1} + \frac{d^2.\text{F}(z)}{dz^2}.\frac{h^2}{1.2}\ldots$$

which is Taylor's series. Taylor's theorem is therefore a particular case of Lagrange's, which, therefore, also includes Maclaurin's.

(60.) *Cor.* 2. If $x = 1$,

$$u = \text{F}(z) + p.q + \frac{d(p^2q)}{dz}.\frac{1}{1.2} + \frac{d^2(p^3q)}{dz^2}.\frac{1}{1.2.3} + \ldots$$

It was in this form that Lagrange delivered the series.

PROP. XXXII.

LAPLACE'S THEOREM.

(61.) *Given* u $=$ F(y) *and* y $=$ F'[z $+$ xf(y)], *to expand the function* u *in a series of ascending positive and integral powers of* x.

Let $\text{F}\{\text{F}'[z+xf(y)]\} = \text{F}''[z+xf(y)]$. Hence $u = \text{F}''(y')$ and $y' = z + xf(y)$. Hence by Taylor's theorem,

$$u = \text{F}''(z) + \frac{d\text{F}''(z)}{dz}.\frac{xf(y)}{1} + \frac{d^2\text{F}''(z)}{dz^2}.\frac{x^2f(y)^2}{1.2} + \ldots$$

Also $f(y) = f\{\text{F}'[z+xf(y)]\} = f'[z+xf(y)]$. Hence by Taylor's series,

$$f(y) = f'(z) + \frac{df'(z)}{dz} \cdot \frac{xf(y)}{1} + \frac{d^2f'(z)}{dz} \cdot \frac{x^2f(y)^2}{1.2} + \ldots .$$

Let $q, q_{1}, q_{2}, \ldots .$ represent $F''(z)$, $\frac{dF''(z)}{dz}$, $\frac{d^2F''(z)}{dz^2}$ $\ldots .$ and let $v = f(y)$, and let p, p_{1}, p_{2}, represent $f'(z)$ and its successive differential coefficients. Hence the preceding series become

$$u = q + q_{1} \frac{xv}{1} + q_2 \cdot \frac{x^2v^2}{1.2} + q_3 \cdot \frac{x^3v^3}{1.2.3} + \ldots .$$

$$v = p + p_{1} \cdot \frac{xv}{1} + p_2 \cdot \frac{x^2v^2}{1.2} + p_3 \cdot \frac{x^3v^3}{1.2.3} + \ldots .$$

Also

$$v^2 = p^2 + \frac{d(p^2)}{dp} \cdot \frac{xv}{1} + \frac{d^2(p^2)}{dp^2} \cdot \frac{x^2v^2}{1.2} + \ldots .$$

$$v^3 = p^3 + \frac{d(p^3)}{dp} \cdot \frac{xv}{1} + \frac{d^2(p^3)}{dp^2} \cdot \frac{x^2v^2}{1.2} + \ldots .$$

. . . .

. . . .

Hence

$$\begin{aligned} u &= q \\ &+ x(q_{1}p) \\ &+ \frac{x^2}{1.2}(q_2p^2 + 2pq_{1}v) \\ &+ \frac{x^3}{1.2.3}\left(q_3p^3 + 3q_2 \cdot \frac{d(p^2)}{dp}v + 3q_{1}p_2v^2\right) \\ &+ \frac{x^4}{1.2.3.4}\left\{q_4 \cdot p^4 + 4q_3 \cdot \frac{d(p^3)}{dp} \cdot v + 6q_2 \cdot \frac{d^2(p^2)}{dp^2} \cdot v^2 + 4q_{1}p_3 \cdot v^3\right\} \end{aligned}$$

. . . .

This series must be equivalent to that of Maclaurin, which gives

$$u = A_0 + A_{1} \frac{x}{1} + A_2 \frac{x^2}{1.2} + A_3 \cdot \frac{x^3}{1.2.3} + \ldots .$$

and therefore the corresponding coefficients must become equal on the condition $x = 0$. In this case $v, v^2, v^3, \ldots .$

become p, p^2, p^3. . . . . Hence we obtain the following equations:

$A_0 = q = F''(z)$,

$A_1 = q_1 p = f'(z) \cdot \frac{dF''(z)}{dz}$,

$$A_2 = q_2 p^2 + 2p q_1 p = \frac{d \cdot \frac{p^2 dq}{dz}}{dz} = \frac{d . f'(z)^2 . \frac{dF''(z)}{dz}}{dz},$$

$$A_3 = q_3 p^3 + 3q_2 . \frac{d(p^2)}{dp} . p + 3 q_1 p_2 p^2 = \frac{d^2 . \frac{p^3 dq}{dz}}{dz^2} = \frac{d^2 . f'(z)^3 . \frac{dF''(z)}{dz}}{dz^2}$$

$$A_4 = q_4 p^4 + 4q_3 . \frac{d(p^3)}{dp} . p + 6q_2 . \frac{d^2(p^2)}{dp^2} . p^2 + 4 q_1 p_3 p^3 =$$

$$\frac{d^3 . f'(z)^4 . \frac{dF''(z)}{dz}}{dz^3}.$$

. . . .

. . . .

Making these substitutions, the result is

$$u = F''(z) + f'(z) . \frac{dF''(z)}{dz} \cdot \frac{x}{1} + \frac{d . f'(z)^2 . \frac{dF''(z)}{dz}}{dz} \cdot \frac{x^2}{1.2}$$

$$+ \frac{d^2 . f'(z)^3 . \frac{dF''(z)}{dz}}{dz^2} \cdot \frac{x^3}{1.2.3} + \frac{d^3 . f'(z)^4 . \frac{dF''(z)}{dz}}{dz^3} \cdot \frac{x^4}{1.2.3.4} \cdot \ldots$$

The nth term of this series being

$$\frac{d^{n-2} . f'(z)^{n-1} . \frac{dF''(z)}{dz}}{dz^{n-2}} \cdot \frac{x^{n-1}}{1.2.3 \ldots (n-1)}$$

Lagrange's theorem is evidently a particular case of this. For in this theorem $F''[z + xf(y)]$ is considered as a function of another function of z, z scil. $F\{F'[z + xf(y)]\}$, and Lagrange's is the particular case where

$$F'[z + xf(y)] = z + xf(y).$$

Laplace has extended this theorem to functions of several variables.

This generalization, however, is not suited to so elementary a treatise as the present.

The preceding proofs of the series of Lagrange and Laplace are taken from the notes of the Cambridge translation of Lacroix, in which the student will find many useful applications of them.

SECTION VI.

Praxis in the development of functions.

PROP. XXXIII.

(62.) *To expand* (x + h)^m *in a series according to the powers of* h.

Let $u = x^m$ and $u' = (x + h)^m$. By differentiation we obtain (39.),

$$\frac{du}{dx} = mx^{m-1}, \frac{d^2u}{dx^2} = m.(m-1).x^{m-2},$$

$$\frac{d^3u}{dx^3} = m(m-1)(m-2)x^{m-3}, \frac{d^4u}{dx^4} = m(m-1)(m-2)(m-3)x^{m-4},$$

. . . .

Hence by Taylor's theorem,

$$u' = x^m + mx^{m-1}.h + \frac{m(m-1)}{1.2}x^{m-2}.h^2 + \frac{m(m-1)(m-2)}{1.2.3}x^{m-3}h^3$$

. . . .

the nth term being

$$\frac{m(m-1)(m-2)\dots m-(n-2)}{1.2.3\dots(n-1)}.x^{m-(n-1)}.h^{n-1}.$$

As the value of m is not restricted, this example contains *the binomial theorem* in its most general state.

PROP. XXXIV.

(63.) *To expand* a^x *in a series of powers of* x.

Let $u = a^x$ ∵ (41.),

$$\frac{du}{dx} = ku, \ \frac{d^2u}{dx^2} = k^2u,$$

and, in general, $\frac{d^nu}{dx^n} = k^nu$. When $x = 0$, $u = 1$. Hence the successive coefficients of Maclaurin's series are $1, k, k^2, \ldots . k^n$, from which it follows that

$$a^x = 1 + \frac{kx}{1} + \frac{k^2x^2}{1.2} + \frac{k^3x^3}{1.2.3} + \frac{k^4x^4}{1.2.3.4} + \ldots .$$

(64.) *Cor.* 1. If $x = \frac{1}{k}$, ∵ $kx = 1$, hence

$$a^{\frac{1}{k}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{\&c.} \ldots$$

this being a converging series, we can approximate indefinitely to its value. Its value continued to seven places of decimals, is 2·7182818. By (21.) it appears that this is the hyperbolic base. Let it be e ∵ $a^{\frac{1}{k}} = e$, ∵ $a = e^k$. Assuming the logarithms $la = kle$, ∵ $k = \frac{la}{le}$, hence, since a and e are known, k is known.

If a be the base of a system of logarithms $le = \frac{1}{k}$. Hence it appears that the modulus of a given system is the logarithm of the hyperbolic base in the given system.

Also $k = l'a$ because $l'e = 1$. Hence the modulus of any system is the reciprocal of the hyperbolic logarithm of the base of the system.

The logarithms of the same number (y) in different systems are as their moduli. For $e^{l'y} = a^{ly}$, a being the base

of the system. Taking the logarithms relatively to the base a, $le \,.\, l'y = ly$; since the number y is given, $l'y$ is constant, therefore $ly \propto le$; that is, the logarithm of a given number is proportional to the modulus of the system.

Hence being given the logarithms of any one system, we can find the corresponding logarithms in any other system whose modulus is given.

(65.) *Cor.* 2. If $a = e \therefore k = 1$, and the series becomes

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} \cdots$$

(66.) *Cor.* 3. If in this series x become successively $+ x\sqrt{-1}$ and $- x\sqrt{-1}$, and the results be added and subtracted, we find

$$e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left\{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} \cdots\right\}$$

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1} \cdot \left\{\frac{x}{1} - \frac{x^3}{1.2.3}\right.$$

$$\left. + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} \cdots \right\}$$

(67.) *Cor.* 4. If in the series for a^x, x become mx,

$$a^{mx} = 1 + \frac{kmx}{1} + \frac{k^2m^2x^2}{1.2} + \frac{k^3m^3x^3}{1.2.3} \cdots$$

Hence it follows that

$$\left\{1 + \frac{kx}{1} + \frac{k^2x^2}{1.2} + \frac{k^3x^3}{1.2.3} \cdots\right\}^m = 1 + \frac{kmx}{1} + \frac{k^2m^2x^2}{1.2} + \frac{k^3m^3x^3}{1.2.3} \cdots$$

PROP. XXXV.

(68.) *To expand the function* l(x + h) *in a series of powers of* h.

Let $u = lx$ and $u' = l(x + h)$. Substituting for the coefficients in Taylor's series their values determined in (42.), the result is

$$u' = u + le\left\{\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} \dots\right\}$$

(69.) *Cor.* 1. Hence we find

$$l(x+h) - lx = le\left\{\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} \dots\right\}$$

When h is small compared with x, this series converges rapidly, and therefore serves, when the logarithm of one number is known, to compute the logarithms of a series which varies by a very small difference.

(70.) *Cor.* 2. If in this series $x = 1$, it becomes

$$l(1+h) = le\left\{\frac{h}{1} - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} \dots\right\}$$

which, when h is negative, becomes

$$l(1-h) = le\left\{-\frac{h}{1} - \frac{h^2}{2} - \frac{h^3}{3} - \frac{h^4}{4} \dots\right\}$$

Hence by subtraction,

$$l\left(\frac{1+h}{1-h}\right) = 2le\left\{\frac{h}{1} + \frac{h^3}{3} + \frac{h^5}{5} \dots\right\}$$

(71.) *Cor.* 3. If $\frac{1+h}{1-h} = 1 + \frac{z}{n}$, $\because h = \frac{z}{2n+z}$.

Hence the last series becomes

$$l(n+z) - ln = 2le\left\{\frac{z}{2n+z} + \tfrac{1}{3}\left(\frac{z}{2n+z}\right)^3 + \tfrac{1}{5}\left(\frac{z}{2n+z}\right)^5 + \&c.\right\}$$

this gives the logarithm of $n+z$ when that of n is known. Let $n = 1$, and $z = 1$, and $le = 1$, hence

$$l'.2 = 2\left\{\frac{1}{3} + \frac{1}{3.3^2} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \dots\right\}$$

this rapidly converges, and therefore gives the hyperbolic logarithm of 2 to any required degree of accuracy. For higher numbers it is still more rapidly convergent.

The modulus may be obtained by calculating the logarithm of the same number in the Neperian or hyperbolic system, and in the system which we wish to adopt.

The function lx cannot be expanded in a series of positive

powers of the variable x. For the first term of Maclaurin's series, being what lx becomes when $x = 0$, is infinite. See (57.).

(72.) *Cor.* 4. If in the series

$$l(1+h) = le\left\{\frac{h}{1} - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} \cdots\right\}$$

h be changed into h^{-1}, it becomes

$$l.\frac{1+h}{h} = l(1+h) - lh = le\left\{\frac{h^{-1}}{1} - \frac{h^{-2}}{2} + \frac{h^{-3}}{3} - \frac{h^{-4}}{4} \cdots\right\}$$

Subtracting the latter from the former, the result is

$$lh = le\left\{\frac{h^1 - h^{-1}}{1} - \frac{h^2 - h^{-2}}{2} + \frac{h^3 - h^{-3}}{3} - \frac{h^4 - h^{-4}}{4} \cdots\right\}$$

PROP. XXXVI.

(73.) *To express the sine and cosine of an arc in a series of powers of the arc itself.*

Let $u = \sin.x$ and $u' = \sin.(x+h)$. By substituting in Taylor's series the values of the differential coefficients given in (43.), we find

$$\sin.(x+h) = \sin.x + \cos.x \cdot \frac{h}{1} - \sin.x \cdot \frac{h^2}{1.2} - \cos.x \cdot \frac{h^3}{1.2.3} + \sin.x \cdot \frac{h^4}{1.2.3.4} + \cos.x \cdot \frac{h^5}{1.2.3.4.5} \cdots$$

Arranging this by the factors $\sin.x$, $\cos.x$, we obtain

$$\sin.(x+h) = \sin.x\left\{1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \frac{h^6}{1.2.3.4.5.6} \cdots\right\} + \cos.x\left\{\frac{h}{1} - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} \cdots\right\}$$

But by trigonometry,

$$\sin.(x+h) = \sin.x\cos.h + \sin.h\cos.x.$$

Since the value of h is independent of x, the equations must hold for all values of x; hence

$$\sin.h = \frac{h}{1} - \frac{h^3}{1.2.3} + \frac{h^5}{1.2.3.4.5} \cdots$$

$$\cos.h = 1 - \frac{h^2}{1.2} + \frac{h^4}{1.2.3.4} - \frac{h^6}{1.2.3.4.5.6} \cdots$$

These series might be also deduced by Maclaurin's theorem, and thence might be obtained by the preceding investigation the trigonometrical formula

$$\sin.(x \pm h) = \sin.x \cos.h \pm \sin.h \cos.x.$$

(74.) *Cor.* 1. Since by (66.), we have

$$e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left\{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} \cdots\right\}$$

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1}\left\{\frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} \cdots\right\}$$

And by the series found in this proposition,

$$\cos.x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} \cdots$$

$$\sin.x = \frac{x}{1} - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} \cdots$$

It follows that

$$2\cos.x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}},$$

$$2\sqrt{-1}.\sin.x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}},$$

and hence

$$\cos.x \pm \sqrt{-1}\sin.x = e^{\pm x\sqrt{-1}}.$$

(75.) *Cor.* 2. Hence also,

$$\cos.mx \pm \sqrt{-1}\sin.mx = e^{\pm mx\sqrt{-1}},$$

$$\because \cos.mx \pm \sqrt{-1}\sin.mx = (\cos.x \pm \sqrt{-1}\sin.x)^m.$$

Also, if $e^{x\sqrt{-1}} = z$, it follows that when

$$2\cos.x = z + \frac{1}{z}, \because 2\cos.mx = z^m + \frac{1}{z^m}, \text{ and}$$

$$2\sqrt{-1}.\sin.mx = z^m - \frac{1}{z^m}.$$

(76.) *Cor.* 3. By division of the results of Cor. 1, we find

$$\sqrt{-1}\ \tan.x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1}.$$

(77.) *Cor.* 4. If in the series found in (72.) $e^{x\sqrt{-1}}$ be substituted for h, and the equation divided by $2\sqrt{-1}$, (le being supposed $= 1$), the result is

$$\frac{x}{2} = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}} - \tfrac{1}{2}\cdot\frac{e^{2x\sqrt{-1}} - e^{-2x\sqrt{-1}}}{2\sqrt{-1}}$$
$$+ \tfrac{1}{3}\frac{e^{3x\sqrt{-1}} - e^{-3x\sqrt{-1}}}{2\sqrt{-1}} \ldots\ldots$$

Making here the substitutions suggested by Cor. 1, we find

$$\frac{x}{2} = \frac{\sin.x}{1} - \frac{\sin.2x}{2} + \frac{\sin.3x}{3} - \frac{\sin.4x}{4} \ldots\ldots$$

PROP. XXXVII.

(78.) *To express a circular arc in a series of powers of its sine.*

Let $u = \sin.^{-1} x$. Substituting for the coefficients in Maclaurin's series what the differential coefficients found in (46.) become when $x = 0$, the result is

$$u = \frac{\sin.u}{1} + \frac{1^2.\sin.^3 u}{1.2.3} + \frac{1^2.3^2.\sin.^5 u}{1.2.3.4.5} + \frac{1^2.3^2.5^2.\sin.^7 u}{1.2.3.4.5.6,7}\ \&c.$$

(79.) *Cor.* 1. If $u = 30^0$, $\because \sin.u = \frac{1}{2}$, $\therefore$

$$\pi = 6\left\{\tfrac{1}{2} + \tfrac{1}{8}\cdot\frac{1}{1.2.3} + \tfrac{1}{32}\cdot\frac{3^2}{1.2.3.4.5} + \ldots\right\}$$

See Geometry (375.).

PROP. XXXVIII.

(80.) *To expand the tangent of an arc in a series of powers of the arc itself.*

Let $u = \tan.x$. Substituting in Maclaurin's series for the coefficients the values of the differential coefficients of this function found in (44.), the result is

$$\tan.x = \frac{x}{1} + \frac{2x^3}{1.2.3} + \frac{2^4.x^5}{1.2.3.4.5} + \ldots.$$

PROP. XXXIX.

(81.) *To expand a circular arc in a series of powers of its tangent.*

Let $u = \tan.^{-1} x$. Substituting in Maclaurin's series the values which the differential coefficients found in (47.) assume when $x = 0$, the result is

$$u = \frac{\tan.u}{1} - \frac{\tan.^3 u}{3} + \frac{\tan.^5 u}{5} - \frac{\tan.^7 u}{7} \ldots.$$

(82.) *Cor.* 1. If $u = \frac{\pi}{4} \because \tan.u = 1, \therefore$

$$\pi = 4\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} \ldots.\}$$

This series is not sufficiently convergent for the purpose of computing the circumference. One may, however, be deduced, which will be sufficiently convergent. See Geometry, vol. i. Art. (375.).

PROP. XL.

(83.) *To express the cotangent of a circular arc in a series of powers of the arc itself.*

Let $u = \cot.x$. In this case the first term of Maclaurin's series becomes infinite. Let $u = x^k z = \cot.x, \therefore z = x^{-k} \cot.x$.

If k be assumed > 0, $x = 0$ renders z infinite. Therefore let $k = -1$. $\therefore z = \frac{x \cos.x}{\sin.x}$.

Substituting for $\cos.x$ and $\sin.x$ their developments obtained in (73.),

$$z = \frac{1 - \frac{1}{1.2}x^2 + \frac{1}{1.2.3.4}x^4 + \dots}{1 - \frac{1}{1.2.3}x^2 + \frac{1}{1.2.3.4.5}.x^4 - \dots}.$$

Hence $z, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \dots$ become $1, 0, -\frac{2}{3}, \dots$ when $x = 0$. Therefore

$$z = 1 - \frac{x^2}{3} - \frac{x^4}{3^2.5} \dots$$

whence we find

$$\frac{z}{x} = \cot.x = x^{-1} - \frac{x}{3} - \frac{x^3}{3^2.5} - \frac{2x^5}{3^2.5.7} - \frac{x^7}{3^3.5^2.7} \dots$$

This process fails in giving the law of the series.

PROP. XLI.

(84.) *To express the value of* u *in* $mu^3 - ux = m$ *in a series of powers of* x.

By differentiating we obtain the values of $\frac{du}{dx}, \frac{d^2u}{dx^2} \dots$ which, when $x = 0$, become $\frac{1}{3m}, 0, -\frac{2}{27m^3} \dots$ and it is evident that when $x = 0$, $u = 1$, $\therefore$ by Maclaurin's series

$$u = 1 + \frac{x}{3m} - \frac{x^3}{3^4m^3} + \frac{x^4}{3^5m^4} - \dots$$

PROP. XLII.

(85.) *To express the value of* u *in the equation* $mu^3 - x^3u - mx^3 = 0$ *in a series of descending powers of* x.

Let $zx^3 = 1$, $\therefore x^3 = \frac{1}{z}$, $\therefore mu^3z - u - m = 0$. The successive differential coefficients of u with respect to z being found, and their values when $z = 0$ substituted for the coefficients in Maclaurin's series, give

$$u = -m - m^4z - 3m^7z^2 - 12m^{10}z^3 + 55m^{13}z^4 \dots$$

$$\text{or } u = -m - m^4x^{-3} - 3m^7x^{-6} - 12m^{10}x^{-9} + 55m^{13}x^{-12} \text{ \&c.}$$

PROP. XLIII.

(86.) *Given* $\text{F}(x + h) + \text{F}(x - h) = \text{F}(x)\text{F}(h)$, *to find the form of the function.*

Let $u = \text{F}(x)$, and $u' = \text{F}(x + h)$, and $u_{,} = \text{F}(x - h)$. By Taylor's theorem,

$$u' = u + \frac{du}{dx} \cdot \frac{h}{1} + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1.2} + \frac{d^3u}{dx^3} \cdot \frac{h^3}{1.2.3} + \frac{d^4u}{dx^4} \cdot \frac{h^4}{1.2.3.4} \dots$$

$$u_{,} = u - \frac{du}{dx} \cdot \frac{h}{1} + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1.2} - \frac{d^3u}{dx^3} \cdot \frac{h^3}{1.2.3} + \frac{d^4u}{dx^4} \cdot \frac{h^4}{1.2.3.4} + \dots$$

adding and dividing by u,

$$\frac{u' + u_{,}}{u} = 2\left\{1 + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1.2} \cdot \frac{1}{u} + \frac{d^4u}{dx^4} \cdot \frac{h^4}{1.2.3.4} \cdot \frac{1}{u} \dots\right\}$$

But $u' + u_{,} = \text{F}(x)\text{F}(h) = u\text{F}(h)$, hence

$$\text{F}(h) = 2\left\{1 + \frac{d^2u}{dx^2} \cdot \frac{h^2}{1.2} \cdot \frac{1}{u} + \frac{d^4u}{dx^4} \cdot \frac{h^4}{1.2.3.4} \cdot \frac{1}{u} + \dots\right\}$$

But $\text{F}(h)$ being independent of x, it follows that

$$\frac{d^2u}{dx^2} \cdot \frac{1}{u}, \quad \frac{d^4u}{dx^4} \cdot \frac{1}{u}, \quad \frac{d^6u}{dx^6} \cdot \frac{1}{u},$$

are constant quantities. Let

$$\frac{d^2u}{dx^2} \cdot \frac{1}{u} = b, \because \frac{d^2u}{dx^2} = bu.$$

By successive differentiation,

$$\frac{d^4u}{dx^4} = b \cdot \frac{d^2u}{dx^2} = b^2u, \because \frac{d^4u}{dx^4} \cdot \frac{1}{u} = b^2,$$

$$\frac{d^6u}{dx^6} = b \cdot \frac{d^4u}{dx^4} = b^3u, \because \frac{d^6u}{dx^6} \cdot \frac{1}{u} = b^3.$$

. . . .

Hence we find

$$\text{F}(h) = 2\left\{1 + \frac{bh^2}{2} + \frac{b^2h^4}{2.3.4} + \frac{b^3h^6}{2.3.4.5.6} \cdots\right\}$$

or substituting for b the constant $-a^2$,

$$\text{F}(h) = 2\left\{1 - \frac{a^2h^2}{2} + \frac{a^4h^4}{2.3.4} - \frac{a^6h^6}{2.3.4.5.6} \cdots\right\}$$

Hence by (73.), this gives

$$\text{F}(h) = 2\cos.ah.$$

It is upon this theorem that *Poisson* founds his proof of the composition of force. (Mecanique, tom, I. p. 15).

PROP. XLIV.

(87.) *To determine the* m*th power of a root of the equation* $xy^n + a - y = 0$, x *and* a *being considered as known.*

Let $u = y^m$, and by the given equation $y = a + xy^n$. Comparing this with Lagrange's theorem,

$$u = \text{F}(y) = y^m,\ z = a,\ f(y) = y^n.$$

Hence $\text{F}(z) = a^m, f(z) = a^n, \dfrac{d\text{F}(z)}{dz} = ma^{m-1}$, &c.

$$y^m = a^m + ma^{m+n-1}.x + \frac{m(m+2n-1)}{1.2}a^{m+2n-2}.x^2 + \cdots$$

which is the development required.

SECTION VII.

Of the limits of series.

PROP. XLV.

(88.) *In any series composed of ascending and positive powers of* h, *a value may be assigned to* h, *so small, that any proposed term may be made to exceed the sum of all that follow it.*

Let $u' = \text{F}(x + h)$, and this being expanded, let

$$u' = u + \text{A}h^a + \text{B}h^b + \text{C}h^c \cdots$$

$a, b, c \cdots$ being in an increasing order.

Let

$$\text{S} = \text{M} + \text{N}h^{n-m} + \text{O}h^{o-m} \cdots$$

$$\text{S}h^m = \text{M}h^m + \text{N}h^n + \text{O}h^o \cdots$$

Therefore

$$u' = u + \text{A}h^a + \text{B}h^b \cdots \text{L}h^l + \text{S}h^m,$$

$$u' = u + \text{A}h^a + \text{B}h^b \cdots h^l(\text{L} + \text{S}h^{m-l}).$$

Since $m > l$, h may be obviously assumed so small that $\text{S}h^{m-l}$ may be indefinitely diminished, and may therefore be rendered less than L. In this case then $\text{L}h^l > \text{S}h^m$, that is, the term $\text{L}h^l$ is greater than the sum of those which succeed it.

(89.) *Cor.* Hence by assuming h sufficiently small, $u' - u$ will take the sign of $\text{A}h^a$, and if a have an even numerator, $u' - u$ will take the sign of A, and will be consequently the same whether h be $+$ or $-$.

PROP. XLVI.

(90.) *To determine the effect which the increase of the variable* x = a *to* x = a + h *produces upon the function.*

Let $u' = \mathrm{F}(x+h)$,

$$u' - u = \frac{du}{dx} \cdot \frac{h}{1} + sh^2.$$

h may be assumed so small, that $\frac{du}{dx} > sh^2$, consequently $u' - u$ will have the sign of $\frac{du}{dx}$. Hence, if the first differential coefficient be positive, the function increases, and if it be negative, the function diminishes. Thus the state of the function for all values of the variable may be determined by finding the roots of the equation $\frac{du}{dx} = 0$. If therefore a and $a+h$ be between two roots of this equation, $\frac{du}{dx}$ does not change its sign between those values of x, and the function increases or decreases according as $\frac{du}{dx}$ is positive or negative.

PROP. XLVII.

(91.) *To determine the limits of the error arising from assuming the first term, the first two terms, or any number of successive terms of the development of* f(x+h), *by Taylor's theorem, for the whole value of* f(x + h).

Let $\frac{du}{dx} = f'(x)$. In this function let x be supposed gradually to increase from x to $x+h$, h being taken of any

finite value. While x varies between these limits, the function $f'(x)$ suffers a corresponding variation; let x' and x'' be the values of x, which, between the proposed limits, render $f'(x)$ greatest and least. The quantities

$$f'(x+h)-f'(x''),\quad f'(x')-f'(x+h)$$

are both positive. These are the differential coefficients of

$$f(x+h)-f(x)-f'(x'')\cdot h,\quad f(x)+f'(x')\cdot h-f(x+h),$$

h being taken as variable. Hence it follows by (90.), that these quantities must increase from x to $x+h$. Now, since they are both $=0$ when $h=0$, it follows that they must be both positive between the proposed limits, and therefore

$$f(x+h)>f(x)+f'(x'')\cdot h, \text{ and } <f(x)+f'(x')\cdot h.$$

If h be negative, the contrary happens. Hence $f(x+h)-f(x)$ is a number included between the values of $f'(x')\cdot h$ and $f'(x'')\cdot h$. If therefore the first term of the development of $f(x+h)$ be taken for the whole value, the error will be greater than $f'(x'')\cdot h$, and less than $f'(x')\cdot h$.

In order to determine the error produced by assuming

$$u+\frac{du}{dx}\frac{h}{1}+\frac{d^2u}{dx^2}\frac{h^2}{1.2}$$

for $u'=f(x+h)$, let $\dfrac{d^2u}{dx^2}=f''(x)$, and, as before, let the greatest and least values of $f''(x)$ from x to $x+h$ be $f''(x')$ and $f''(x'')$. The quantities

$$f''(x+h)-f''(x''),\quad f''(x')-f''(x+h),$$

are positive, and therefore the quantities

$$f'(x+h)-f'(x)-f''(x'')\cdot h,\quad f'(x)+f''(x')\cdot h-f'(x+h),$$

of which they are the differential coefficients, h being the variable, must continually increase between the proposed limits, and must therefore be positive, since they vanish when $h=0$. Hence the quantities

$$f(x+h) - f(x) - f'(x)\frac{h}{1} - f''(x'')\frac{h^2}{1.2},$$

$$f(x) + f'(x)\frac{h}{1} - f''(x')\frac{h^2}{1.2} - f(x+h),$$

of which the former are the differential coefficients, must be both positive. Hence

$$f(x+h) > u + \frac{du}{dx}\frac{h}{1} + f''(x'')\frac{h^2}{1.2},$$

$$f(x+h) < u + \frac{du}{dx}\frac{h}{1} + f''(x')\frac{h^2}{1.2}.$$

Hence if $u + \frac{du}{dx}\frac{h}{1}$ be taken as the value of $f(x+h)$, the error is comprised between the limits $f''(x'')\frac{h^2}{1.2}$ and $f''(x')\frac{h^2}{1.2}$.

In general, therefore, if n terms of the series be taken for the whole value of u' or $f(x+h)$, the error is comprised between the limits of the greatest and least values of

$$\frac{d^n \cdot u}{dx^n} \cdot \frac{h^n}{1.2 \cdots n},$$

x being supposed to vary in the function $\frac{d^n u}{dx^n}$ from x to $x+h$. It is, however, to be understood, that there are no values of x comprised between x and $x+h$, which render the function u or any of its differential coefficients infinite; in other words, it is necessary that there should be no values of x between these limits which furnish exceptions to Taylor's series.

Ex. 1. Let $u = a^x$, $\because \frac{d^n u}{dx^n} = k^n a^x$, and if $u' = a^{x+h}$,

$$\because \frac{d^n u'}{dx^n} = k^n a^{x+h}.$$

The greatest and least value of $\frac{d^n u'}{dx^n}$ between x and $x+h$ are

the values corresponding to $x + h$ and x themselves. Therefore $f^{(n)}(x') = k^n a^{x+h}$ and $f^{(n)}(x'') = k^n a^x$. Hence the limits of error are included between the quantities

$$a^x \cdot \frac{k^n h^n}{1.2.3 \dots n},$$

$$a^{x+h} \cdot \frac{k^n h^n}{1.2.3 \dots n}.$$

Ex. 2. Let $u = l(x + h)$. The limits of error in this case are

$$\frac{h^n}{n} \cdot \frac{1}{x^n},$$

$$\frac{h^n}{n} \cdot \frac{1}{(x+h)^n}.$$

PROP. XLVIII.

(92.) *Two series ascending by the powers of the same quantity* (h), *being given, to determine the limit of their ratio,* h *being indefinitely diminished.*

Let the two series be

$$\text{s} = \text{A}h^a + \text{B}h^b + \text{C}h^c \dots$$
$$\text{s}' = \text{A}'h^{a'} + \text{B}'h^{b'} + \text{C}'h^{c'} \dots$$

When h is diminished without limit, the limit of

$\frac{\text{s}}{\text{s}'}$ is $\frac{\text{A}h^a}{\text{A}'h^{a'}}$. For

$$\frac{\text{s}}{\text{s}'} = \frac{h^a(\text{A} + \text{B}h^{b-a} + \text{C}h^{c-a} \dots)}{h^{a'}(\text{A}' + \text{B}'h^{b'-a'} + \text{C}'h^{c'-a'} \dots)}.$$

When $h = 0$, the factors within the parenthesis become A and A$'$, and therefore in the limit

$$\frac{\text{s}}{\text{s}'} = \frac{\text{A}0^a}{\text{A}'0^{a'}}.$$

If $a = a'$, the limit is $\frac{\text{A}}{\text{A}'}$. If $a > a'$, the limit is 0; and if $a < a'$, the limit is infinity.

(93.) *Cor.* Hence if $a = a'$, a value may be assigned to h so small as to render $S >$ or $< S'$ according as A is $>$ or $< A'$, and S will continue $>$ or $< S'$ for all values between that assigned value of h and 0.

If $a > a'$, a value may be assigned to h so small as to render $S < S'$, and S will continue less than S' for all lesser values of h.

If $a < a'$, a value may be assigned to h so small as to render $S > S'$, and S will continue greater for all lesser values of h.

If any number of successive terms of the two series commencing from the first be respectively equal, that is, if $A = A'$, $a = a'$, $B = B'$, $b = b'$, &c. then the relative values of S and S', h being supposed to be indefinitely diminished, may be determined by the first pair of corresponding terms of the series which are not equal, in the same manner as above. Thus, if the terms of the series be respectively equal as far as Lh^l and $L'h^{l'}$, inclusive, let the common values of the series thus far be s. Then

$$\frac{S-s}{S'-s} = \frac{Mh^m + Nh^n \cdots\cdots}{M'h^{m'} + N'h^{n'} \cdots\cdots}.$$

If $m = m'$, $\therefore S - s$ is $>$ or $< S' - s$, or S is $>$ or $< S'$, according as M is $>$ or $< M'$, h being assumed sufficiently small. And all that has been observed before applies here *mutatis mutandis*.

SECTION VIII.

Of the differentiation of functions of several variables.

(94.) The functions which have been subjected to investigation in the preceding sections have been supposed to

be composed of one variable quantity, connected by some given or supposed relation with constant quantities, or if more than one variable has been introduced, they have been always understood to be connected by some condition, which being expressed by an equation, an elimination might be effected, by which we might finally arrive at a function of a single variable. We shall, however, now proceed to consider a more extensive class of functions, namely, those whose variation depends on the variation of several quantities, which are independent of one another.

As the variation of functions of several variables is produced by the several variations of each of the variables, there are as many differential coefficients of the first order as there are variables independent of each other. In general, when u is a function of several variables x, x', x'' . . . the differential coefficient determined by considering u as a function of the variable x alone, all the other variables being supposed to be constant, is called the *partial differential coefficient* of the first order of u differentiated with respect to x, and this coefficient is expressed $\frac{du}{dx}$, as if the function u were a function of x alone. This differentiation being continued, a series of successive *partial* differential coefficients will be found which are expressed, $\frac{d^2u}{dx^2}$, $\frac{d^3u}{dx^3}$ . . . . $\frac{d^nu}{dx^n}$, as if u were a function of x alone. In the use of these symbols, therefore, their true meaning should be carefully attended to, and the student should be cautious not to use them as if they referred to the entire variation of the function u, but only to that part of it which depends upon the variation of x. In like manner $\frac{du}{dx'}$ $\frac{d^2u}{dx'^2}$ . . . . $\frac{d^nu}{dx'^n}$ are the partial differential coefficients depending on the variation of

x' alone; and in a similar way the partial differential coefficients depending on the variation of any other variable may be expressed.

There are therefore as many partial differential coefficients of this kind of any proposed order as there are variables.

Besides the species of differential coefficients above mentioned, there are also others obtained in a different manner. The partial differential coefficients of the first order are themselves functions of the original variables. The differentiation of the function $\frac{du}{dx}$ has been continued as a function of x. But as it is a function of the other variables x', x'' .... as well as of x, it is susceptible of differentiation with respect to any one them. If $\frac{du}{dx}$ be considered as a function of x', all the other variables being considered constant, it may be differentiated. The differential coefficient resulting from this process would, according to the system of notation used in functions of a single variable, be $\frac{d\cdot\left(\frac{du}{dx}\right)}{dx'}$. To avoid the complexity of these symbols, it is, however, expressed thus, $\frac{d^2u}{dxdx'}$, which signifies, therefore, the differential coefficient obtained by differentiating the function u with respect to x, and again differentiating the result of that operation with respect to x'.

If the function u had been first differentiated with respect to x', and next with respect to x, the differential coefficient would be expressed thus, $\frac{d^2u}{dx'dx}$.

The differential coefficient obtained by two successive differentiations with respect to x being $\frac{d^2u}{dx^2}$; if this be consi-

dered as a function of x', and as such be differentiated, the coefficient resulting from this operation is expressed thus, $\frac{d^3u}{dx^2dx'}$. If, on the other hand, the function u be first differentiated with respect to x', and the result $\frac{du}{dx'}$ be twice differentiated with respect to x, the differential coefficient resulting from this operation is expressed thus, $\frac{d^3u}{dx'dx^2}$.

In general, if the function u be differentiated successively m times with respect to the variable x, and the result n times with respect to the variable x', the differential coefficient which results from these operations is expressed thus,

$$\frac{d^{m+n}u}{dx^m dx'^n}.$$

But if the function be first differentiated n times with respect to x', and then m times with respect to x, the resulting coefficient is expressed thus,

$$\frac{d^{n+m}u}{dx'^n dx^m}.$$

(95.) To explain this notation generally, let u be a function of the variables $x', x'', x''' \dots x^{(n)}$, and let it be differentiated m' times successively with respect to x', and the result of that operation m'' times successively with respect to x'' and so on.

The differential coefficient which results from this system of operations is expressed thus,

$$\frac{d^{m'+m''\dots m^{(n)}}\dots u}{dx'^{m'}.dx''^{m''}\dots .dx^{(n)m^{(n)}}}.$$

If in the function u, the several variables $x', x'', \dots x^{(n)}$ be supposed to receive the increments $h', h'', \dots h^{(n)}$, and the function to become u', let it be supposed to be expanded

in a series according to the positive and integral powers of the increments h', h'', ... $h^{(n)}$. The first term, which is independent of the increments, must be the original function u; for it is what the series which is equal to u' becomes when the several increments $= 0$. Let the series be

$$\begin{aligned} u' = u &+ \text{A}'h' + \text{A}''h'' + \cdots\cdot \text{A}^{(n)}.h^{(n)}, \\ &+ \text{B}'h'^2 + \text{B}''h''^2 + \cdots\cdot \text{B}^{(n)}.h^{(n)2}, \\ &+ \text{C}'h'h'' + \text{C}''h''h''' + \cdots\cdot \text{C}^{(n)}h^{(n-1)}h^{(n)}, \\ &+ \text{D}'h'^3 + \text{D}''h''^3 + \cdots\cdot \text{D}^{(n)}h^{(n)3}, \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \quad [1]. \end{aligned}$$

The sum of the terms of this series which involve the first powers of the increments is called the *total differential* of the function u, and each of these terms are called *partial differentials*. Thus,

$$du = \text{A}'h' + \text{A}''h'' \cdots\cdot \text{A}^{(n)}h^{(n)},$$

or, as it is more usually expressed,

$$du = \text{A}'dx' + \text{A}''dx'' \cdots\cdot \text{A}^{(n)}dx^{(n)}.$$

If in the series [1] h'', h''' .... $h^{(n)}$ be supposed severally $= 0$, the series becomes

$$u' = u + \text{A}'h' + \text{B}'h'^2 + \text{D}'h'^3 \cdots\cdot$$

This is equivalent to supposing x' alone variable, and x'', x''', ... $x^{(n)}$ constant. Hence it follows that $\dfrac{du}{dx'} = \text{A}'$.

In like manner $\dfrac{du}{dx''} = \text{A}''$, and so on. Hence

$$du = \frac{du}{dx'}dx' + \frac{du}{dx''}dx'' \cdots\cdot \frac{du}{dx^n}dx^{(n)}.$$

In the use of this notation, it should be observed that $\dfrac{du}{dx'}dx'$, does not signify a division and subsequent multiplication of du by dx'. In this case $\dfrac{du}{dx'}$ is not the representative of a quote, but represents a function of the several

variables x', x'', ... $x^{(n)}$, derived from the primitive function u in the manner already explained; and $\frac{du}{dx'}\cdot dx'$ signifies this function multiplied by dx'. This product, as has been already observed, is called *a partial differential* of u.

PROP. XLIX.

(96.) *It is required to express a function* $u' = F(x' + h', x'' + h'' \dots x^{(n)} + h^{(n)})$ *in a series of positive and integral powers of the quantities* $h', h'', \dots h^{(n)}$, *the quantities* x', x'', ... $x^{(n)}$ *being independent variables.*

We shall first consider the case in which there are but two independent variables x', x''. In this case

$$u' = F(x' + h', x'' + h'').$$

As the exponents of the quantities h', h'', in the required series, are positive integers, the series arranged by the powers of h' must have the form

$$u' = \left.\begin{matrix} A \\ +A'h'' \\ +A''h''^2 \\ +A'''h''^3 \\ \dots \\ \dots \end{matrix}\right\} \left.\begin{matrix} +B \\ +B'h'' \\ +B''h''^2 \\ +B'''h''^3 \\ \dots \\ \dots \end{matrix}\right\} \cdot h' \left.\begin{matrix} +C \\ +C'h'' \\ +C''h''^2 \\ +C'''h''^3 \\ \dots \\ \dots \end{matrix}\right\} \cdot h'^2 \left.\begin{matrix} +D \\ +D'h'' \\ +D''h''^2 \\ +D'''h''^3 \\ \dots \\ \dots \end{matrix}\right\} \cdot h'^3 \dots [1]$$

If u' be considered a function of x' only, x'' being treated as a constant quantity, it may be expanded in a series of powers of h' by Taylor's theorem, and if $u'' = F(x', x'' + h'')$, this expansion will be

$$u' = u'' + \frac{du''}{dx'}\cdot\frac{h'}{1} + \frac{d^2u''}{dx'^2}\cdot\frac{h'^2}{1.2} + \frac{d^3u''}{dx'^3}\cdot\frac{h'^3}{1.2.3} + \dots$$

This series must be indentical with the former, independently of h', and therefore

$$u'' = \text{A} + \text{A}'h'' + \text{A}''h''^2 + \text{A}'''h''^3 + \cdots\cdots \quad [2],$$

$$\frac{1}{1}\cdot\frac{du''}{dx'} = \text{B} + \text{B}'h'' + \text{B}''h''^2 + \text{B}'''h''^3 + \cdots\cdots \quad [3],$$

$$\frac{1}{1.2}\cdot\frac{d^2u''}{dx'^2} = \text{C} + \text{C}'h'' + \text{C}''h''^2 + \text{C}'''h''^3 + \cdots\cdots \quad [4],$$

$$\frac{1}{1.2.3}\cdot\frac{d^3u''}{dx'^3} = \text{D} + \text{D}'h'' + \text{D}''h''^2 + \text{D}'''h''^3 \cdots\cdots \quad [5].$$

. . . .

If $u = \text{F}(x'x'')$, and this be considered as a function of x'', x' being considered constant, u'' may be expanded by Taylor's theorem, the result is

$$u'' = u + \frac{du}{dx''}\cdot\frac{h''}{1} + \frac{d^2u}{dx''^2}\cdot\frac{h''^2}{1.2} + \frac{d^3u}{dx''^3}\cdot\frac{h''^3}{1.2.3} + \cdots \quad [6].$$

If u and its partial differential coefficients $\frac{du}{dx''}$, $\frac{d^2u}{dx''^2}$, &c. be considered as functions of x', this series, differentiated successively, gives

$$\frac{du''}{dx'} = \frac{du}{dx'} + \frac{d^2u}{dx''dx'}\cdot\frac{h''}{1} + \frac{d^3u}{dx''^2dx'}\cdot\frac{h''^2}{1.2} + \frac{d^4u}{dx''^3dx'}\cdot\frac{h''^3}{1.2.3}\cdots [7],$$

$$\frac{d^2u''}{dx'^2} = \frac{d^2u}{dx'^2} + \frac{d^3u}{dx''dx'^2}\cdot\frac{h''}{1} + \frac{d^4u}{dx''^2dx'^2}\cdot\frac{h''^2}{1.2} + \frac{d^4u}{dx''^3dx'^2}\cdot\frac{h''^3}{1.2.3}\cdots\cdots \quad [8],$$

$$\frac{d^3u''}{dx'^3} = \frac{d^3u}{dx'^3} + \frac{d^4u}{dx''dx'^3}\cdot\frac{h''}{1} + \frac{d^5u}{dx''^2dx'^3}\cdot\frac{h''^2}{1.2} + \frac{d^6u}{dx''^3dx'^3}\cdot\frac{h''^3}{1.2.3}\cdots\cdots \quad [9].$$

. . . .

These series [6], [7], [8], [9], &c. must be identical respectively with [2], [3], [4], [5], &c. independently of h''. Hence

$$\text{A} = u,\ \text{A}' = \frac{du}{dx''},\ \text{A}'' = \frac{d^2u}{dx''^2}\cdot\frac{1}{2},\ \text{A}''' = \frac{d^3u}{dx''^3}\cdot\frac{1}{1.2.3}\cdots\cdots$$

$$\text{B} = \frac{du}{dx'},\ \text{B}' = \frac{d^2u}{dx''dx'},\ \text{B}'' = \frac{d^3u}{dx''^2dx'}\cdot\frac{1}{1.2},\ \text{B}''' = \frac{d^4u}{dx''^3dx'}\cdot\frac{1}{1.2.3}\cdots$$

$$\text{C} = \frac{d^2u}{dx'^2},\ \text{C}' = \frac{d^3u}{dx''dx'^2},\ \text{C}'' = \frac{d^4u}{dx''^2dx'^2}\cdot\frac{1}{1.2},\ \text{C}''' = \frac{d^5u}{dx''^3dx'^2}\cdot\frac{1}{1.2.3}\cdots$$

$$\text{D} = \frac{d^3u}{dx'^3},\ \text{D}' = \frac{d^4u}{dx''dx'^3},\ \text{D}'' = \frac{d^5u}{dx''^2dx'^3}\cdot\frac{1}{1.2},\ \text{D}''' = \frac{d^6u}{dx''^3dx'^3}\cdot\frac{1}{1.2.3}\cdots$$

. . . .

Hence the expansion of u', arranged according to the powers of h' and h'', is

$$u' = u + \left.\begin{array}{l} \dfrac{du}{dx'}\cdot\dfrac{h'}{1} \\ +\dfrac{du}{dx''}\cdot\dfrac{h''}{1} \end{array}\right\} + \left.\begin{array}{l} \dfrac{d^2u}{dx'^2}\cdot\dfrac{h'^2}{1.2} \\ +\dfrac{d^2u}{dx''dx'}\cdot\dfrac{h'h''}{1} \\ +\dfrac{d^2u}{dx''^2}\cdot\dfrac{h''^2}{1.2} \end{array}\right\} + \left.\begin{array}{l} \dfrac{d^3u}{dx'^3}\cdot\dfrac{h'^3}{1.2.3} \\ +\dfrac{d^3u}{dx''^2dx'}\cdot\dfrac{h''^2h'}{1.2} \\ +\dfrac{d^3u}{dx''dx'^2}\cdot\dfrac{h''h'^2}{1.2} \\ +\dfrac{d^3u}{dx''^3}\cdot\dfrac{h''^3}{1.2.3} \end{array}\right\} + \cdots [10]$$

As the variables in the function $u = \text{F}(x'x'')$ are not distinguished by any particular condition, this series will still represent u', if x' and h' be changed into x'' and h'', and *vice versa*. This change gives

$$u' = u + \left.\begin{array}{l} \dfrac{du}{dx''}\cdot\dfrac{h''}{1} \\ +\dfrac{du}{dx'}\cdot\dfrac{h'}{1} \end{array}\right\} + \left.\begin{array}{l} \dfrac{d^2u}{dx''^2}\cdot\dfrac{h''^2}{1.2} \\ +\dfrac{d^2u}{dx'dx''}\cdot\dfrac{h''h'}{1} \\ +\dfrac{d^2u}{dx'^2}\cdot\dfrac{h'^2}{1.2} \end{array}\right\} + \left.\begin{array}{l} \dfrac{d^3u}{dx''^3}\cdot\dfrac{h''^3}{1.2.3} \\ +\dfrac{d^3u}{dx'^2dx''}\cdot\dfrac{h'^2h''}{1.2} \\ +\dfrac{d^3u}{dx'dx''^2}\cdot\dfrac{h'h''^2}{1.2} \\ +\dfrac{d^3u}{dx'^3}\cdot\dfrac{h'^3}{1.2.3} \end{array}\right\} + \cdots [11]$$

As these series both represent u', independently of the values of h' and h'', the corresponding coefficients must be equal. Hence

$$\frac{d^2u}{dx'dx''} = \frac{d^2u}{dx''dx'},$$

$$\frac{d^3u}{dx''dx'^2} = \frac{d^3u}{dx'^2dx''},$$

$$\frac{d^4u}{dx''^3dx'} = \frac{d^4u}{dx'dx''^3},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and, in general,

$$\frac{d^{m+n} \cdot u}{dx'^m dx''^n} = \frac{d^{n+m} \cdot u}{dx''^n dx'^m}.$$

It follows, therefore, that if any function of two variables be differentiated m times for one of the variables, and n times for the other, the resulting differential coefficient will be the same in whatever order the differentiations may have been performed.

The analogy of each successive column of the developments [10] and [11] to the terms of an expanded binomial is quite obvious. If the quantities dx' dx'' be transferred from the denominators of the coefficients to the denominators of the powers of h' and h'', the successive vertical columns may be represented thus,

$$\frac{du}{1}\left\{\frac{h'}{dx'} + \frac{h''}{dx''}\right\},$$

$$\frac{d^2u}{1.2}\left\{\frac{h'}{dx'} + \frac{h''}{dx''}\right\}^2$$

$$\frac{d^3u}{1.2.3}\left\{\frac{h'}{dx'} + \frac{h''}{dx''}\right\}^3.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^nu}{1.2.3 \cdots n}\left\{\frac{h'}{dx'} + \frac{h''}{dx''}\right\}^n.$$

And, therefore, if $s = \frac{h'}{dx'} + \frac{h''}{dx''}$, the series will become

$$u' = u + \frac{sdu}{1} + \frac{s^2d^2u}{1.2} + \frac{s^3d^3u}{1.2.3} \cdot \quad \cdot \quad \cdot \quad \cdot \qquad [12]$$

the nth term being

$$\frac{s^nd^nu}{1.2.3 \cdots n}.$$

It should be however remembered, that when s^n is sup-

posed to be developed, and the resulting terms $\frac{d^n u}{dx'^n}$, $\frac{d^n u}{dx'^{n-1}\,dx''}$ &c. found, these symbols are not meant to represent the operations of involution by which in this way they are produced, but express the results of the successive differentiations as explained in (95).

If u' be considered as a function of three or more variables, we shall, by continuing the same process, find a similar result; and if

$$s = \frac{h'}{dx'} + \frac{h''}{dx''} \cdots \frac{h^{(n)}}{dx^{(n)}},$$

the series [12] still represents u'.

It also follows, on exactly the same principles as in the case of only two variables, that the value of each differential coefficient is not affected by the order in which the successive differentiations are performed. For example,

$$\frac{d^{m'+m''+m'''\cdots\cdots}\,.\,u}{dx'^{m'}dx''^{m''}dx'''^{m'''}\cdots\cdot} = \frac{d^{m'''+m'+m''\cdots\cdots}\,.\,u}{dx'''^{m'''}dx'^{m'}dx''^{m''}\cdots\cdot}.$$

If the several arbitrary quantities dx', dx'', $\cdots\cdot dx^{(n)}$ be assumed equal to the increments h', h'', $\cdots\cdot h^{(n)}$, $\because s = 1$, and the series [12] becomes

$$u' = u + \frac{du}{1} + \frac{d^2u}{1.2} + \frac{d^3u}{1.2.3} \cdots$$

which has been applied to functions of one variable in (53).

PROP. L.

(97.) *To differentiate a quantity composed of several functions of several independent variables united by addition or subtraction, the differentials of the component functions being given.*

Let $u = x' + x'' \cdots\cdot x^{(n)}$. By the principles already established,

$$\frac{du}{dx'} = 1, \frac{du}{dx''} = 1, \text{\&c.}$$

Hence

$$du = dx' + dx'' \dots + dx^{(n)},$$

which extends the result of (17.) to functions of several independent variables.

PROP. LI.

(98.) *To differentiate the product of several independent variables.*

Let $u = x'x'' \dots x^{(n)}$. By the rules already laid down, if the partial differentials be found, and added, the equation

$$du = \frac{du}{dx'}dx' + \frac{du}{dx''}\cdot dx'' \dots \frac{du}{dx^{(n)}}\cdot dx^{(n)}$$

becomes in this case,

$$du = x''x''' \dots x^{(n)}dx' + x'x''' \dots x^{(n)}dx'' \dots x'x'' \dots x^{(n)}dx^{(n)},$$

which is the same as the result of (22.), where x', x'', ... are not independent variables, but all functions of a common variable.

In a similar way the result of (23.) may be extended to fractions, of which the numerator and denominator are products of independent variables.

(99.) The following examples will illustrate the principles on which functions of two variables are differentiated.

Ex. 1. Let $u = x^m y^n$, ∵

$$\frac{du}{dx} = mx^{m-1}\cdot y^n,$$

$$\frac{du}{dy} = nx^m y^{n-1}.$$

Hence

$$du = mx^{m-1}y^n\cdot dx + nx^m y^{n-1}\cdot dy = x^{m-1}\cdot y^{n-1}(mydx + nxdy).$$

Ex. 2. Let $u = l\cdot\tan.\dfrac{x}{y}$. If $z = \dfrac{x}{y}$ and $z' = \tan.z$, then $u = lz'$, and by (16.)

$$\frac{du}{dz} = \frac{du}{dz'}\cdot\frac{dz'}{dz}.$$

But $\dfrac{du}{dz'} = \dfrac{1}{z'}$, and $\dfrac{dz'}{dz} = \dfrac{1}{\cos.^2 z}$. Hence

$$\frac{du}{dz} = \frac{\cot.z}{\cos.^2 z} = \frac{1}{\sin.z\cos.z}.$$

But since $z = \dfrac{x}{y}$, $\because$

$$dz = \frac{ydx - xdy}{y^2}.$$

Hence

$$du = \frac{ydx - xdy}{y^2\sin.\dfrac{x}{y}\cos.\dfrac{x}{y}}.$$

Ex. 3. Let $u = \tan.^{-1}\sin.^{-1}\dfrac{x}{y}$. As before, let $z = \dfrac{x}{y}$ and $z' = \sin.^{-1}z$, $\because u = \tan.^{-1}z'$. Hence (16.),

$$\frac{du}{dz} = \frac{du}{dz'}\cdot\frac{dz'}{dz}.$$

But $\dfrac{du}{dz'} = \dfrac{1}{1+z'^2}$ and $\dfrac{dz'}{dz} = \dfrac{1}{\sqrt{1-z^2}}$. Hence

$$\frac{du}{dz} = \frac{1}{(1+z'^2)\sqrt{1-z^2}}.$$

But since $z = \dfrac{x}{y}$, $\because$

$$dz = \frac{ydx - xdz}{y^2}.$$

Hence

$$du = \frac{ydx - xdy}{y^2(1+(\sin.^{-1}\dfrac{x}{y})^2)\cdot\sqrt{1-\dfrac{x^2}{y^2}}} = \frac{ydx - xdy}{y(1+(\sin.^{-1}\dfrac{x}{y})^2)\sqrt{y^2-x^2}}.$$

SECTION IX.

The differentiation of equations of several variables.

(100.) When an equation $F(xy) = 0$, involving two variables, is given, either variable y may be considered as a function of the other x. The resolution of the equation for y would change it to the form

$$y = f(x).$$

In this state the function y might be differentiated by the rules already given for the differentiation of functions of one variable, and thence the value of the successive differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}$, found.

This method, however, would in general be of no practical use, as it would require the general resolution of equations. It will be therefore necessary to find a method of determining the successive differential coefficients of y with respect to x, without resolving the proposed equation for y.

(101.) For this purpose let x and y be first supposed to be independent variables, and let $u = F(xy)$, the values of y and x not being *necessarily* limited by the condition $u = 0$. If y and x become $y + k$ and $x + h$, the function becomes $u' = F[(x + h), (y + k)]$, and by the equation [11] (96.)

$$u' = u + \left.\begin{matrix} \frac{du}{dx} \cdot \frac{h}{1} \\ + \frac{du}{dy} \cdot \frac{k}{1} \end{matrix}\right\} + \left.\begin{matrix} \frac{d^2u}{dx^2} \cdot \frac{h^2}{1.2} \\ + \frac{d^2u}{dxdy} \cdot \frac{hk}{1} \\ + \frac{d^2u}{dy^2} \cdot \frac{k^2}{1.2} \end{matrix}\right\} + \dots$$

If in the equation $y = f(x)$, x become $x + h$ and y become y', we have by Taylor's series

$$k = y' - y = \frac{dy}{dx} \cdot \frac{h}{1} + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} + \frac{d^3y}{dx^3} \cdot \frac{h^3}{1.2.3} \cdot \cdot \cdot \cdot$$

Making this substitution for k in the series already found, and arranging the result by the dimensions of h, it assumes the form,

$$u' = u + \text{A}' \cdot \frac{h}{1} + \text{A}'' \cdot \frac{h^2}{1.2} + \text{A}''' \cdot \frac{h^3}{1.2.3} + \cdot \cdot \cdot \cdot [1].$$

Where

$$\text{A}' = \frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = \left(\frac{du}{dx} \cdot dx + \frac{du}{dy} \cdot dy\right) \frac{1}{dx},$$

$$\text{A}'' = \frac{d^2u}{dx^2} \cdot \frac{1}{2} + \frac{d^2u}{dx \cdot dy} \cdot \frac{dy}{dx} + \frac{d^2u}{dy^2} \cdot \frac{dy^2}{dx^2} \cdot \frac{1}{2},$$

$$\text{or A}'' = \left(\frac{d^2u}{dx^2} \cdot dx^2 + 2 \cdot \frac{d^2u}{dxdy} \cdot dxdy + \frac{d^2u}{dy^2} dy^2\right) \cdot \frac{1}{2dx^2},$$

$$\text{A}''' = \left(\frac{d^3u}{dx^3} \cdot dx^3 + 3 \cdot \frac{d^3u}{dx^2dy} dx^2dy + 3 \frac{d^3u}{dxdy^2} dxdy^2 + \frac{d^3u}{dy^3} dy^3\right) \cdot \frac{1}{1.2.3.dx^3}.$$

And in general the series of coefficients of the powers of dx and dy in each term is evident from their analogy to the coefficients of an expanded binomial.

Let the variables x and y be now restricted so as always to satisfy the equation $u = 0$, so that whatever be the value of h, the condition $u' = 0$ must be fulfilled. In this case the series [1] must $= 0$ independently of h; hence its several coefficients must separately $= 0$, which gives the equations

$$u = 0,$$

$$\frac{du}{dx} dx + \frac{du}{dy} dy = 0,$$

$$\frac{d^2u}{dx^2} dx^2 + 2 \frac{d^2u}{dxdy} \cdot dxdy + \frac{d^2u}{dy^2} dy^2 = 0,$$

$$\frac{d^3u}{dx^3}dx^3 + 3\frac{d^3u}{dx^2dy}\cdot dx^2dy + 3\frac{d^3u}{dxdy^2}dxdy^2 + \frac{d^3u}{dy^3}dy^3 = 0.$$

. . . .

The first of these equations is only a repetition of $F(xy) = 0$. The second, however, determines the value of the differential coefficient $\frac{dy}{dx}$, if the functions $\frac{du}{dx}$ and $\frac{du}{dy}$ be known. Let these be A and B, ∴

$$Ady + Bdx = 0,$$

$$\therefore \frac{dy}{dx} = -\frac{B}{A}.$$

Hence it follows, that "to find the first differential coefficient of an implicit function y, given by an equation of two variables x and y, the equation must be differentiated as a function of two independent variables, and the *total differential* being equated with zero, will determine the sought differential coefficient."

(102.) In a similar way it may be shown that an equation of any number of variables may be treated as a function of the variables, and differentiated. Let $u = F(x', x'' \cdots x^{(n)}) = 0$, by differentiation we obtain

$$\frac{du}{dx'}dx' + \frac{du}{dx''}\cdot dx'' + \cdot \cdot \cdot \cdot \frac{du}{dx^{(n)}}dx^{(n)} = 0.$$

This is called the *total differential* of the proposed equation.

The *partial differential* equations may be obtained by considering the given equation successively as a function of each combination of two variables. This process will give as many partial differential equations as there are different combinations of two variables in the primitive equation, and each of these equations will determine a partial differential coefficient of one of the variables as a function of another. As however the differentials of the variables severally enter

as multipliers of all the terms of these equations, any one of them may be deduced from the others.

(103.) To return to the equations of two variables, the differential coefficient $\frac{dy}{dx}$ being expressed by p, the differential equation becomes

$$\text{A}p + \text{B} = 0 \ [1].$$

In this case A and B being functions of the variables x and y, this may be treated as an equation of three variables, x, y, and p, and being expressed by u', its differential is

$$\frac{du'}{dx} \cdot dx + \frac{du'}{dy} \cdot dy + \frac{du'}{dp} dp = 0 \ [2].$$

Since $dp = \frac{d^2y}{dx}$, it is evident that this equation, combined with the first differential equation, will determine the differential coefficient $\frac{d^2y}{dx^2}$ as a function of the variables x, y.

Hence "to obtain the second differential coefficient, the first differential equation must be differentiated, considering x, y, and $\frac{dy}{dx}$ as variables."

Again the equation [2] being differentiated, considering x, y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as variable, will give a third equation, which, combined with the other two, will determine the third differential coefficient $\frac{d^3y}{dx^3}$.

Thus, in general, "the equations which determine the successive differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2} \cdots \frac{d^ny}{dx^n}$, of an implicit function given by an equation of two variables, are deduced by successive differentiations, each differential coefficient being considered as an additional variable."

SECTION X.

Of the effect of particular values of the variable upon a function, and its differential coefficients.

(104.) A function is in general rendered either positive or negative by the real values which may be assigned to the variable. There are, however, four states of the function which are attended with peculiar circumstances, and which require some examination. Certain particular values of x may render the function, or its differential coefficients, $1^0 = 0$, $2^0 = \frac{0}{0}$, 3^0 imaginary, 4^0 infinite. We shall consider these four cases first in explicit, and next in implicit functions.

PROP. LII.

(105.) *To determine the values of the successive differential coefficients of a function* (u) *which correspond to any particular value* (a) *of the variable* (x), *which renders the function or any of its differential coefficients* $= 0$.

1°. Let $x = a$ render the function itself $= 0$. By the principles of Algebra, it follows that $x - a$, or some positive power of it must be a factor of u; so that u must be of the form $u = \text{P}(x - a)^m$, m being a positive integer or fraction, and P being a function of x not divisible by $(x - a)$, or any power of it.

From the process of differentiation it appears that $(x - a)^{m-1}$, $(x - a)^{m-2}$, &c. are factors of the successive differential coefficients of u. Let these coefficients be u', u'', ... $u^{(n)}$, they must be of the forms

$$u' = P'(x-a)^{m-1},$$
$$u'' = P''(x-a)^{m-2},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$u^{(n)} = P^{(n)}(x-a)^{m-n},$$

where P', P'' .... are quantities not divisible by any power of $(x-a)$.

If m be an integer, these successive differential coefficients will $= 0$ when $x = a$ as far as the $(m-1)$th inclusive; but the mth differential coefficient will be of the form

$$u^{(m)} = P^{(m)}(x-a)^{m-m} = P^{(m)},$$

which not being divisible by any power of $(x-a)$, will not vanish when $x = a$. The same may be observed of the differential coefficients which succeed the mth.

It is plain that if $m = 1$, the function vanishes, but none of its differential coefficients do.

If m be a fraction, let n be the next integer below it, and $\therefore n+1$ the next above it. In this case the differential coefficients as far as the nth inclusive vanish with the function, and those that succeed it all become infinite. This is evident from considering that $m-n$ is positive, and $m-(n+1)$ negative.

If m be a proper fraction, then $n = 0$; and in this case all the differential coefficients are infinite.

2°. Let $x = a$ render any proposed differential coefficient $= 0$. If the first differential coefficient which it renders $= 0$ be of the nth order, it follows that

$$u^{(n)} = P^{(n)}(x-a)^m,$$

$P^{(n)}$ not being divisible by a power of $x-a$.

In this case it may be proved by the process already used, that when m is an integer, the differential coefficients from the nth to the $(n+m-1)$th inclusive vanish when $x = a$, and those which succeed them do not. If m be a fraction between l and $l+1$, then the differential coefficients from the

nth to the $(n + l)$th vanish, and the succeeding coefficients become infinite.

If m be a proper fraction, then all the coefficients after the nth become infinite.

PROP. LIII.

(106.) *Given a function which vanishes when* $x = a$, *to determine the highest power of* $(x - a)$, *which divides the function.*

Let $u = F(x)$, which vanishes when $x = a$. It is measured by $(x - a)^z$ to determine z. Let the function be differentiated until a differential coefficient $u^{(n)}$ be found which does not vanish when $x = a$. This coefficient will be either finite or infinite. If it be finite, the value of z is an integer, and $= n$. If it be infinite, the value of z is a fraction, whose value is between the integers n and $n - 1$. To determine it, let $u^{(n-1)}$ be divided by such a fractional power k of $a - x$, that the quote $\frac{u^{(n-1)}}{(a-x)^k}$ may be finite when $x = a$. Then the exponent of the sought power will be $n + k$. This is manifest from the last proposition.

PROP. LIV.

(107.) *To determine the true value of a function which a particular value of* x *renders* $\frac{0}{0}$, *or infinite.* fraction

That the first may take place, it is necessary that the numerator and denominator be both functions of x, which vanish when $x = a$, and which therefore have factors of the form $(x - a)^z$.

Let the function then be

$$u = \frac{F(x)}{F'(x)};$$

and let the highest power of $(x - a)$ which divides one be z, and the other z'. The function may therefore be expressed thus,

$$u = \frac{P(x-a)^z}{P'(x-a)^{z'}}.$$

The values of z and z' are to be determined as in the last proposition.

If $z > z'$, $u = 0$. If $z < z'$, u is infinite. If $z = z'$, $u = \frac{P}{P'}$.

Hence it appears that the method of proceeding to determine the value of the function is, to differentiate both the numerator and denominator until a differential coefficient of each be found, which does not vanish when $x = a$. Let this coefficient be of the nth order for the numerator, and of the mth for the denominator.

Then

1°. If $n > m$, the function is $= 0$.

2°. If $n < m$, the function is infinite, as well as all its differential coefficients.

3°. If $n = m$, the nth differential coefficient in each term of the fraction may be either finite or infinite. This presents four cases, *First,* If it be infinite in the numerator, and finite in the denominator; in this case z is a fraction less than n and $z' = n$; hence the function is infinite.

Secondly. If it be infinite in the denominator, and finite in the numerator, then $z = n$ and z' is a fraction less than n, therefore the value of the function is 0.

Thirdly. If both be finite; in this case the value of the function is a fraction whose numerator and denominator are the differential coefficients themselves. For let $x + h$ be

substituted for x in both numerator and denominator, and the results developed, we find

$$U = \frac{F(x) + A'.\frac{h}{1} + A''.\frac{h^2}{1.2} + \cdots\cdot}{F'(x) + B'.\frac{h}{1} + B''.\frac{h^2}{1.2} + \cdots\cdot},$$

where A', A'', &c. B', B'', &c. are the successive differential coefficients.

Substituting a for x, the functions and their successive coefficients vanish as far as the nth differential coefficient, which is by hypothesis finite in both numerator and denominator. Hence the function becomes

$$U = \frac{A^{(n)}.\frac{h^n}{1.2\cdots n} + A^{(n+1)}.\frac{h^{n+1}}{1.2\cdots n+1}}{B^{(n)}.\frac{h^n}{1.2\cdots n} + B^{(n+1)}.\frac{h^{n+1}}{1.2\cdots n+1}} \cdots\cdots$$

Dividing both terms by h^n, and supposing $h = 0$, we find

$$u = \frac{A^{(n)}}{B^{(n)}},$$

which is a fraction whose numerator and denominator are the first differential coëfficients which remain finite when $x = a$.

Fourthly. If the first differential coefficients which do not vanish be of the same (nth) order, and both become infinite when $x = a$. In that case z and z' are both fractions between the integers $n - 1$ and n. The values of the fractions may be determined as in (106.); and if they be equal, both terms of the fraction being divided by the common power of $x - a$, the result will be its true value. If $z > z'$, $u = 0$, and if $z < z'$, u is infinite. Or the value may be determined thus. In both numerator and denominator let $x + h$ be substituted for x, and the results expanded according to increasing powers of h by the ordinary rules of

Algebra, or by the method explained in (55.); for in this case the series of Taylor does not apply (55.); and let the result be

$$\text{U} = \frac{\text{A}h^{a} + \text{A}'h^{a'} + \text{A}''h^{a''} \dots}{\text{B}h^{b} + \text{B}'h^{b'} + \text{B}''h^{b''} \dots}.$$

The exponents $a, a', a'' \dots b, b', b'' \dots$ being arranged in an increasing order. If $a > b$, the fraction becomes

$$\text{U} = \frac{\text{A}h^{a-b} + \text{A}'h^{a'-b} + \text{A}''h^{a''-b} \dots}{\text{B} + \text{B}'h^{b'-b} + \text{B}''h^{b''-b} \dots}.$$

It is evident that all the exponents in this case are positive, and therefore by substituting a for x, and making $h = 0$, we find $u = 0$.

If, however, $a < b$, the fraction becomes

$$\text{U} = \frac{\text{A} + \text{A}'h^{a'-a} + \text{A}''h^{a''-a} \dots}{\text{B}h^{b-a} + \text{B}'h^{b'-a} + \text{B}''h^{b''-a} \dots}.$$

Making $x = a$, and $h = 0$, this becomes infinite.

If $a = b$, by dividing both numerator and denominator by h^{a},

$$\text{U} = \frac{\text{A} + \text{A}'h^{a'-a} + \text{A}''h^{a''-a} \dots}{\text{B} + \text{B}'h^{b'-a} + \text{B}''h^{b''-a} \dots}.$$

Substituting a for x, and supposing $h = 0$,

$$u = \frac{\text{A}}{\text{B}}.$$

If in the product of two functions of x one factor become infinite when $x = a$, and the other 0, it can be reduced to the form $\frac{0}{0}$, and therefore its value may be found by the preceding rules.

Let $u = \text{F}(x) \times \text{F}'(x)$, and let $\text{F}'(x)$ be infinite and $\text{F}(x) = 0$ when $x = a$.

If we suppose $f(x) = \dfrac{1}{\text{F}'(x)}$, then if $x = a$, $f(x) = 0$.

But $u = \dfrac{\text{F}(x)}{f(x)}$, which becomes $\frac{0}{0}$ when $x = a$.

If $u = \frac{F(x)}{F'(x)}$, and $x = a$, render both numerator and denominator infinite, the value of u may be found by the same rules: for let $f(x) = \frac{1}{F(x)}, f'(x) = \frac{1}{F'(x)}$,

$$\therefore u = \frac{f'(x)}{f(x)}, \text{ which becomes } \tfrac{0}{0} \text{ when } x = a.$$

Also, if $u = F(x) - F'(x)$, and these functions become infinite when $x = a$, let $F(x) = \frac{1}{f(x)}$, and $F'(x) = \frac{1}{f'(x)}$

$$\therefore u = \frac{f'(x) - f(x)}{f(x) f'(x)},$$

which becomes $\frac{0}{0}$ when $x = a$. Hence all combinations of functions of x coming under the preceding forms are regulated by the rules already delivered.

(108.) We shall now proceed to give some examples of the application of these rules.

Ex. 1. Let $u = \frac{x^n - 1}{x - 1}$, to find the value of this when $x = 1$. By differentiating once, we find the first differential coefficients of the numerator and denominator,

$$\frac{nx^{n-1}}{1}.$$

Hence $u = n$.

Ex. 2. Let $u = \frac{F(x)}{F'(x)} = \frac{ax^2 + ac^2 - 2acx}{bx^2 - 2bcx + bc^2}$, to find the value of u when $x = c$. By differentiating

$$\frac{dF(x)}{dx} = 2a(x - c),$$

$$\frac{d \cdot F'(x)}{dx} = 2b(x - c).$$

These both $= 0$ when $x = c$. Differentiating therefore again,

$$\frac{d^2\text{F}(x)}{dx^2} = 2a,$$

$$\frac{d^2\cdot\text{F}'(x)}{dx^2} = 2b,$$

which not being affected by the value of x, give

$$u = \frac{a}{b}.$$

Ex. 3. $u = \dfrac{a^x - b^x}{x}$, to find u when $x = 0$.

$$\frac{d\text{F}(x)}{dx} = a^x la - b^x lb,$$

$$\frac{d\text{F}'(x)}{dx} = 1.$$

Hence $u = la - lb = l\left(\dfrac{a}{b}\right)$.

Ex. 4. $u = \dfrac{1 - \sin.x + \cos.x}{\sin.x + \cos.x - 1}$, to find u when $x = \dfrac{\pi}{2}$.

$$\frac{d\text{F}(x)}{dx} = -\cos.x - \sin.x,$$

$$\frac{d\text{F}'(x)}{dx} = \cos.x - \sin.x.$$

Hence when $x = \dfrac{\pi}{2}$, $u = 1$.

Ex. 5. Let $u = \dfrac{(x^2 - a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}$, to find u when $x = a$.

The value may easily be found in this case by raising both to the power $\frac{2}{3}$, $\because$

$$u^{\frac{2}{3}} = \frac{x^2 - a^2}{x - a} = x + a.$$

Hence $u^{\frac{2}{3}} = 2a$, $\because u = (2a)^{\frac{3}{2}}$.

Ex. 6. Let $u = \dfrac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}}$, to determine u

when $x = a$. In this case, let $(a + h)$ be substituted for x, and the result is

$$u' = \frac{\sqrt{a+h} - \sqrt{a} + \sqrt{h}}{\sqrt{2ah+h^2}}.$$

Developing $(a + h)^{\frac{1}{2}}$ by the binomial theorem, and substituting for it its development

$$u' = \frac{\sqrt{h} + \frac{1}{2}a^{-\frac{1}{2}}\cdot h + \cdots}{\sqrt{h}\cdot\sqrt{2a+h}}.$$

Dividing both terms of this fraction by $\sqrt{h}$, we find

$$u' = \frac{1 + \frac{1}{2}a^{-\frac{1}{2}}\cdot h^{\frac{1}{2}} + \cdots}{\sqrt{2a+h}}.$$

Making $h = 0$, we find

$$u = \frac{1}{\sqrt{2a}}.$$

Ex. 7. Let $u = (1 - x)\tan.\frac{1}{2}(\pi x)$, to find u when $x = 1$. In this case u assumes the form $0 \times \infty$. But

$\tan.\frac{1}{2}(\pi x) = \frac{1}{\cot.\frac{1}{2}(\pi x)}$, $\therefore u = \frac{1-x}{\cot.\frac{1}{2}(\pi x)}$, which becomes $\frac{0}{0}$ when $x = 1$. Applying to this the common rule

$$\frac{d\text{F}(x)}{dx} = -1,$$

$$\frac{d\text{F}'(x)}{dx} = -\frac{\frac{1}{2}\pi}{\sin.^2\frac{1}{2}(\pi x)}.$$

When $x = 1$, $\sin.^2\frac{1}{2}(\pi x) = 1$, $\therefore u = \frac{2}{\pi}$.

Ex. 8. Let $u = \dfrac{\tan.\frac{1}{2}\pi\cdot\frac{x}{a}}{x^2a^{-1}(x^2-a^2)^{-1}}$, to determine the value of u when $x = a$. In this case the fraction becomes $\frac{\infty}{\infty}$.

But since $\tan.\frac{1}{2}\pi \cdot \frac{x}{a} = \frac{1}{\cot.\frac{1}{2}\pi \cdot \frac{x}{a}}$, and

$$x^2a^{-1}(x^2 - a^2)^{-1} = \frac{x^2}{a(x^2 - a^2)}, \because$$

$$u = \frac{a(x^2 - a^2)x^{-2}}{\cot.\frac{1}{2}\pi \cdot \frac{x}{a}},$$

which becomes $\frac{0}{0}$ when $x = a$. Differentiating, we find

$$\frac{d\text{F}(x)}{dx} = 2ax^{-1} - 2a(x^2 - a^2)x^{-3},$$

$$\frac{d\text{F}'(x)}{dx} = - \frac{\frac{1}{2}\pi \cdot}{a \sin.^2\frac{1}{2}\pi \cdot \frac{x}{a}}.$$

Hence when $x = a$, $u = -\frac{4a}{\pi}$.

Ex. 9. Let $u = x\tan.x - \frac{1}{2}\pi\sec.x$, to find u when

$$x = \frac{\pi}{2}.$$

In this case $u = \infty - \infty$. But since

$$\tan.x = \frac{1}{\cot.x}, \ \sec.x = \frac{1}{\cos.x},$$

$$\because u = \frac{x}{\cot.x} - \frac{\pi}{2\cos.x} = \frac{x\sin.x - \frac{1}{2}\pi}{\cos.x};$$

which, when $x = \frac{\pi}{2}$, becomes $\frac{0}{0}$. Differentiating, we find

$$\frac{d\text{F}(x)}{dx} = x\cos.x + \sin.x,$$

$$\frac{d\text{F}'(x)}{dx} = - \sin.x.$$

Hence, when $x = \frac{\pi}{2}$, $u = -1$.

PROP. LV.

(109.) *To determine the conditions under which any differential coefficient of an implicit function becomes* $= 0$, $= \frac{1}{0}$, or $= \frac{0}{0}$.

Let the equation by which y is an implicit function of x be $\text{F}(xy) = 0$, which we shall suppose cleared of irrational quantities; and let the first differential equation be

$$\text{A}dy + \text{B}dx = 0,$$

A and B being each functions of x and y. In order that the value of $\frac{dy}{dx}$ deduced from this equation may be $= 0$, it is necessary that $\text{B} = 0$; but this is not sufficient. It is to be considered that the variables x and y must satisfy the primitive equation $\text{F}(xy) = 0$. Hence it is also necessary that the equation obtained by eliminating one of the variables by means of the equations

$$\text{F}(xy) = 0,$$
$$\text{B} = 0,$$

should have at least one real root which is not infinite. Such a root will determine a corresponding value of the other variable, and this system of values *may* render $\frac{dy}{dx} = 0$. We say, *may* render it so, because there is still a possibility of the contrary. If the system of values thus determined satisfy the equation $\text{A} = 0$, then the differential coefficient becomes $\frac{0}{0}$, and its value must be sought by a method which will be explained hereafter. If, however, the system of values of xy so determined do not satisfy the equation $\text{A} = 0$, then the corresponding value of the first differential coefficient must be 0.

By continuing the process of differentiation any number

of times, and eliminating each differential coefficient, except the last found by the preceding differential equations, an equation will be obtained of the form

$$\text{A}d^n y + \text{B}dx^n = 0.$$

The preceding observations will apply here also. If the values of x, y, determined by the equations

$$\text{F}(xy) = 0,$$
$$\text{B} = 0,$$

be real and not infinite, and do not satisfy the condition $\text{A} = 0$, then $\frac{d^n y}{dx^n} = 0$.

It does not follow that if one differential coefficient $= 0$, all or any of those which succeed it will also $= 0$. For if y be supposed to be eliminated from $\text{B} = 0$ by $\text{F}(xy) = 0$, then B will be a function of x alone; and if a value of x, which satisfies the condition $\text{B} = 0$, be a, then the equation may be expressed under the form

$$\text{C}(x - a)^m = 0.$$

The differential coefficients of this will $= 0$, for $x = a$ as far as the differential coefficient, whose order is marked by the integer next below m, but no further (105).

(110.) In order that any differential coefficient should become infinite, it is necessary that the system of values of x and y determined by the equations

$$\text{F}(xy) = 0,$$
$$\text{A} = 0$$

should be real and not infinite. If this system of values do not satisfy the equation $\text{B} = 0$, the corresponding value of $\frac{d^n y}{dx^n}$ will be infinite.

Any system of values of the variables xy, which renders any differential coefficient infinite, also renders all those which succeed it infinite. For let

$$\text{A}d^n y + \text{B}dx^n = 0$$

be differentiated, the result will be of the form

$$\text{A}d^{n+1}y + \text{C}dx^{n+1} = 0.$$

The same conditions, therefore, which render $\frac{d^ny}{dx^n}$ infinite, also render $\frac{d^{n+1}y}{dx^{n+1}}$ infinite.

(111.) If the value of the first differential coefficient derived from the equation

$$\text{A}dy + \text{B}dx = 0,$$

assume the form $\frac{0}{0}$, the conditions,

$$\text{A} = 0, \text{B} = 0,$$

must be fulfilled, as well as $\text{F}(xy) = 0$. In order, therefore, to determine the possibility of this, let the variables be eliminated by these equations, and the resulting equation will only include constants, and ought to be identically zero, or of the form

$$\text{C} - \text{C} = 0.$$

Having determined whether there be any such systems of values of the variables, it is necessary next to determine the corresponding value of the differential coefficient $\frac{dy}{dx}$. Let $p = \frac{dy}{dx}$, ∴

$$\text{A}p + \text{B} = 0.$$

This being differentiated, gives

$$\text{A}dp + pd\text{A} + d\text{B} = 0.$$

Since A and B are functions of x and y, it follows that $d\text{A}$ and $d\text{B}$ must be of the form

$$\text{C}dy + \text{D}dx,$$

$$\text{C}'dy + \text{D}'dx.$$

Making these substitutions, dividing by dx, and putting p for $\frac{dy}{dx}$, the result will be of the form

$$\text{A} \cdot \frac{dp}{dx} + \text{A}'p^2 + \text{B}'p + \text{C}' = 0.$$

Now by the original conditions $A = 0$, $\because$ the value of p is to be determined by the equation

$$A'p^2 + B'p + C' = 0.$$

The roots of this equation are subject to all the varieties incident on the roots of equations of the second degree. They may be real or impossible, equal or unequal, nothing or infinite, under the usual conditions.

If, however, this equation, like the first, be fulfilled by its coefficients; that is, if the system of values of xy determined by the first three equations satisfy the equations,

$$A' = 0, \; B' = 0, \; C' = 0,$$

then this equation cannot determine the value of p. In this case it will be necessary to differentiate it, considering x, y, and p as variables. The result of this will have the form

$$(2A'p + B') \cdot \frac{dp}{dx} + A''p^3 + B''p^2 + C''p + D'' = 0.$$

But the conditions $A' = 0$, $B' = 0$, render this

$$A''p^3 + B''p^2 + C''p + D'' = 0,$$

which determines the values of p. The same observations may be made here, as before, as to the nature of the roots. Also, if this be fulfilled by its coefficients, it is necessary to differentiate again, which will give an equation of the fourth degree for p.

These conclusions will equally apply to any differential coefficient of an higher order by supposing $p = \frac{d^n y}{dx^n}$.

(112.) It appears, therefore, that when any system of values of the variables renders a differential coefficient $\frac{0}{0}$, that coefficient *may* have several real values corresponding to the same system of values of the variables.

(113.) The converse of this also follows. If for any system of values of the variables, any differential coefficient have more values than one, then that coefficient

derived from the differential equation of the corresponding order must assume the form $\frac{0}{0}$. Since the original equation $F(xy) = 0$ is supposed to be rational with respect to the variables, and since the process of differentiation never introduces radicals, the differential equations must be all rational. Let two values of the differential coefficient of the nth order be p, p'. These being substituted in the differential equation, give

$$Ap + B = 0,$$
$$Ap' + B = 0.$$

Since A and B are rational functions of the variables, they cannot have more values than one for a given system of values of the variables. Therefore, the values of A and B in these two equations must be necessarily the same. Subtracting, we find

$$A(p - p') = 0,$$
$$\therefore A = 0,$$
$$\therefore B = 0,$$
$$\therefore p = \tfrac{0}{0}, p' = \tfrac{0}{0}.$$

(114.) As the state of the function corresponding to such a system of values of the variables as that we have just been considering is attended with circumstances of some importance in Geometry, we shall examine it somewhat more particularly.

Let y' be what y becomes when x becomes $x + h$. By Taylor's series, we find

$$y' = y + A_{\prime} \cdot \frac{h}{1} + A_2 \cdot \frac{h^2}{1.2} + A_3 \cdot \frac{h^3}{1.2.3} \cdots\cdots$$

where $A_{\prime}$, $A_2 \cdots\cdot$ are the differential coefficients.

Now, if for the same values of y and x, $A_{\prime}$ have two unequal values, there will necessarily be two corresponding unequal values of y', whether h be affirmative or negative; but when $h = 0$, the value of y' becomes single, and equal to y. Hence it appears that this circumstance must arise from the particular values of the variables which render $A_{\prime} = 0$,

making a radical vanish in the value of y derivable from the original equation, and yet not making the same radical vanish in $A_{,}$. This follows from the principle, that the roots of an equation can only become equal by a radical disappearing. The possibility of a radical disappearing in a function, and yet reappearing in its differential coefficients, may easily be shown.

Let

$$u = (x - a)\sqrt{b^2 + x^2} + c.$$

When $x = a$, $u = c$, and the radical disappears. But

$$\frac{du}{dx} = \frac{x(x-a)}{\sqrt{b^2+x^2}} + \sqrt{b^2 + x^2}.$$

When $x = a$,

$$\frac{du}{dx} = \sqrt{b^2 + x^2},$$

in which the radical appears. See Geometry Art. (368.), note.

It appears, therefore, that if the first differential coefficient becomes $\frac{0}{0}$, and its values be determined by an equation of the second degree, whose roots are real and unequal, Taylor's series divides itself into two at the second term; thus,

$$y' = y + \begin{cases} A_{,} \cdot \dfrac{h}{1} + A_2 \cdot \dfrac{h^2}{1.2} + \cdots \\ B_{,} \cdot \dfrac{h}{1} + B_2 \cdot \dfrac{h^2}{1.2} + \cdots \end{cases}$$

In like manner, if the value of the differential coefficient were determined by an equation of the third degree, it would divide itself into three, and so on.

If the differential coefficient of the nth order assume the form $\frac{0}{0}$, the series divides itself at the $(n+1)$th term; thus,

$$y' = y + A_{,} \cdot \frac{h}{1} + A_2 \cdot \frac{h^2}{1.2} + \cdots + \begin{cases} A_n \cdot \dfrac{h^n}{1.2 \cdots n} + \cdots \\ B_n \cdot \dfrac{h^n}{1.2 \cdots n} + \cdots \end{cases}$$

And in general the series divides itself into as many different series as there are unequal finite and real values of the differential coefficient.

SECTION XI.

Of maxima and minima.

(115.) Let u be a function of the variable x, and let three values of u corresponding to $x = a - h$, $x = a$, and $x = a + h$, be

$$\begin{aligned} u' &= \mathrm{F}(a - h), \\ u'' &= \mathrm{F}(a), \\ u''' &= \mathrm{F}(a + h). \end{aligned}$$

Def. If a be such a value of x, that for any finite value of h however small, the quantities $u''-u'$ and $u''-u'''$ have the same sign, and continue to have that sign for all values of h between that finite value and 0, then the value u'' is called a *maximum* or *minimum* value of the function according as the common sign of the quantities $u'' - u'$ and $u'' - u'''$ is $+$ or $-$.

(116.) From this, which is a rigorous definition of maxima and minima, it will be perceived that these terms do not necessarily signify the greatest or least value of the function. It is true, that if the function is incapable of unlimited increase or decrease, and therefore has a greatest or least value, this value must be a maximum or minimum, and this case will be found to come within the preceding definition. But on the other hand, the function may have maxima and minima values which are not its greatest or least values, and may even have several maxima and several minima of different values. This will be easily conceived, if, while the variable x is supposed continually to increase

from 0 to infinity, the function be supposed to vary, and in its variation, alternately to increase and decrease, the value of the function, which stands exactly between its increase and decrease, or at which it changes from its increasing state to its decreasing state, is a *maximum;* and that value at which it changes from a decreasing state to an increasing state is a minimum. Upon examining the definition already given, it will be found that these principles are involved in it. An example will probably put the matter in a clearer point of view.

Let $u = b - (x - a)^2$. If $x = 0$, $u = b - a^2$; if b be supposed $> a^2$, this value of u is positive. As x increases from $x = 0$ to $x = a$, the quantity $(x - a)^2$ diminishes from a^2 to 0, and therefore u increases from $u = b - a^2$ to $u = b$. When x becomes $> a$, the quantity $(x - a)^2$ again increases, and therefore u diminishes, and therefore the value $u = b$ stands between the increase and decrease of the function, and is therefore a *maximum.*

Again, let $u = b + (x - a)^2$. In this case, when $x = 0$, $u = b + a^2$, the quantity $(x - a)^2$ decreases from $x = 0$ to $x = a$, for which $u = b$. When x becomes greater than a, u begins to increase; hence in this case $u = b$ is a *minimum* value of the function.

(117.) From the definition of maxima and minima, it follows that the essential characteristic of a maximum is, that it exceeds those values of the function which immediately precede and follow it, while a minimum is less than both these values.

(118.) The general method of determining maxima and minima of functions of a single variable is derived from Taylor's series, except when the values of x come under its exceptions. We shall first consider the cases which do not fall within the exceptions. Let $u'' = \text{F}(x)$, and $u' = \text{F}(x - h)$, $u''' = \text{F}(x + h)$; hence

$$u'' - u' = p' \frac{h}{1} - p'' \frac{h^2}{1.2} + p'''. \frac{h^3}{1.2.3} \cdots$$

$$u'' - u''' = - p'. \frac{h}{1} - p''. \frac{h^2}{1.2} - p''' . \frac{h^3}{1.2.3} \cdots$$

$p', p'', p''' \ldots$ expressing the successive differential coefficients of the function.

Let these series be expressed thus,

$$u'' - u' = h . \left\{ p' - p''. \frac{h}{1.2} + p'''. \frac{h^2}{1.2.3} \cdots \right\}$$

$$u'' - u''' = h . \left\{ -p' - p''. \frac{h}{1.2} - p'''. \frac{h^2}{1.2.3} \cdots \right\}.$$

Such a value may be assigned to h as will render the sum of all the terms of these series which succeed the first less than the first, and therefore the signs of the entire series will be those of their first terms. If the quantity p' be not $= 0$, the value of u'' cannot be either a maximum or minimum; for by assigning a sufficiently small value to h, the sign of $u'' - u'$ will be that of $+ p'$, and $u'' - u'''$ will be that of $-p'$, these signs being different, the value u'' does not come under the definition. In order that u'' should be a maximum or minimum, it is therefore necessary that $p' = 0$, and as p' is a function of x, it follows that no value of x but such as are roots of the equation $p' = 0$, can render the function either a maximum or minimum.

Let it therefore be supposed that a value of x, which is a root of this equation, be substituted for x in the functions u'', p', p'', &c. We shall first suppose that this value of x is not a root of the equation $p'' = 0$. The series by this substitution become

$$u'' - u' = h^2 \left\{ -p'' + p''' \frac{h}{1.2.3} \cdots \right\},$$

$$u'' - u''' = h^2 \left\{ -p'' - p''' \frac{h}{1.2.3} \cdots \right\}.$$

In this case, as before, such a value may be assigned to h

as will render the first term greater than the remainder of each series. Hence the quantities $u'' - u'$ and $u'' - u'''$ will both have the sign of $-p''$, and will have the same sign for every value of h between that and 0. The corresponding value of the function will therefore be a maximum if $p'' < 0$, and a minimum if $p'' > 0$.

If, however, the root of the equation $p' = 0$, which is substituted for x, be also a root of the equation $p'' = 0$, but not a root of $p''' = 0$, the series become

$$u'' - u' = h^3 \left\{ +p''' - p'''' \cdot \frac{h}{1.2.3.4} \cdots \right\},$$

$$u'' - u''' = h^3 \left\{ -p''' - p'''' \cdot \frac{h}{1.2.3.4} \cdots \right\}.$$

In this case, as in the first, a value may be assigned to h such, that $u'' - u'$ shall have the sign of $+ p'''$, and $u'' - u'''$ the sign of $- p'''$, and therefore the corresponding value of the function is not either a maximum or minimum.

If, however, the value of x be also a root of $p''' = 0$, and not of $p'''' = 0$, the function is a maximum or minimum, according as $p'''' < 0$, or > 0, and so on.

Hence we conclude, that in order to determine the maxima and minima values of a function, it is necessary first to find the first differential coefficient (p'). This being, in general, a function of x, determines those values of x which render it $= 0$, or the roots of the equation $p' = 0$. No values of the variable x, which are not exceptions to Taylor's series, can render the function u a maximum or minimum, but such as are found amongst the real roots of this equation. Substitute these roots successively for x in the second differential coefficient p''. Such of them as render $p'' < 0$, being substituted for x in the function u, give maximum values; such as render $p'' > 0$, give minimum values. If, however, any of them render $p'' = 0$, they must be substituted in p'''; and if they render it $>$ or $<$ 0, they will not render the function u either

maximum or minimum; but if they also render $p''' = 0$, they must be substituted in p'''', and so on; and if the first differential coefficient, which they render $>$ or $<$ 0, be of an odd order, they do not give either maxima or minima values of the function; but if it be of an even order, they determine maxima or minima according as they render that differential coefficient negative or positive.

(119.) We shall now consider the maxima and minima values of the function u, which are found among those values which form exceptions to Taylor's series.

Let the values of x which are roots of the equation $\frac{1}{p'} = 0$ be determined. In this case the developments become

$$u'' - u' = \mathrm{A}(-h)^a + \mathrm{B}(-h)^b + \mathrm{C}(-h)^c \ldots .$$
$$u'' - u''' = \mathrm{A}h^a + \mathrm{B}h^b + \mathrm{C}h^c \ldots \ldots .$$

If any of the exponents have an even denominator, the consideration of maxima and minima becomes inapplicable; for, the fraction being in its least terms, the numerator must be odd, therefore one series will be real, and the other imaginary, since the corresponding term is the even root of an odd power; and therefore the value of the function does not come under the definition (115.).

If all the denominators be odd, the numerator of the exponent a may either be odd or even. If it be odd, $(+h)^a$ and $(-h)^a$ will have different signs, and $\therefore$ u'' will be neither a maximum nor minimum.

But if the numerator of a be even, $(+h)^a$ and $(-h)^a$ will be both positive, being the odd root of an even power; and $\therefore$ in this case u'' will be a maximum when A is positive, and a minimum when A is negative.

If a value of x which renders $p' = 0$, render p'' infinite, the developments assume the same forms as in the last case,

and the same observations exactly will apply; and will in general apply if a value of x, which renders the differential coefficients from the first to the nth inclusive $= 0$, render the $(n+1)$th infinite.

(120.) Before the more general investigation of the maxima and minima of functions of several variables, it may be useful to give some examples of the determination of those of functions of a single variable.

Ex. 1. Let $u = ax^3 - bx^2 + x + 9$. Hence

$$\frac{du}{dx} = 3ax^2 - 2bx + 1.$$

The values of x, which render this $= 0$, are

$$x = \frac{b + \sqrt{b^2 - 3a}}{3a},$$

$$x = \frac{b - \sqrt{b^2 - 3a}}{3a}.$$

If $b^2 < 3a$, these values are both impossible, and therefore the function in this case is not capable of a maximum or minimum. But if b^2 be not $< 3a$, let the function be differentiated again, and the result is

$$\frac{d^2u}{dx^2} = 6ax - 2b.$$

Substituting in this the values of x already found,

$$\frac{d^2u}{dx^2} = \pm 2\sqrt{b^2 - 3a}.$$

If $b^2 > 3a$, one value of x renders $\frac{d^2u}{dx^2} > 0$, and the other < 0. Hence in this case, if

$$\frac{b - \sqrt{b^2 - 3a}}{3a}$$

be substituted for x in the function u, the corresponding value will be a maximum; and if

$$\frac{b+\sqrt{b^2-3a}}{3a}$$

be substituted, the corresponding value is a minimum.

If, however, $b^2 = 3a$, and $\because b^2 - 3a = 0$; in this case the value of x determined by $\frac{du}{dx} = 0$ is $x = \frac{1}{b}$, which being substituted for x in

$$\frac{d^2u}{dx^2} = 6ax - 2b,$$

renders $\frac{d^2u}{dx^2}=0$. It will therefore be necessary to differentiate again, which gives

$$\frac{d^3u}{dx^3} = 6a.$$

This not depending on x, and not being $= 0$, the function admits of no maximum or minimum in this case.

Ex. 2. *To divide a number* a *into two parts such, that the product of the* mth *power of one, and the* nth *power of the other, shall be a maximum or minimum.*

If x be one of the parts, and $\because a - x$ the other, the product is

$$u = x^m(a - x)^n,$$

$$\because \frac{du}{dx} = (a - x)^{n-1} . x^{m-1} . \{ma - (m + n)x\},$$

$$\frac{d^2u}{dx^2}=(a-x)^{n-2}.x^{m-2}.\{(ma-(m+n)x)^2-m(a-x)^2-nx^2\}.$$

The values of x, which render $\frac{du}{dx} = 0$, are determined by the equations

$$a - x = 0,$$
$$x = 0,$$
$$ma - (m + n)x = 0,$$

which give

$$x = a,$$
$$x = 0,$$
$$x = \frac{ma}{m+n}.$$

The value $x = a$ renders the second differential coefficient $= 0$; and it is evident that since every differential coefficient of an order inferior to the nth will have $x - a$, or some power of it as a factor, the same value of x will render all these $= 0$. The nth differential coefficient will not have the factor $x - a$, and therefore, in it changing x into a, the result will not be found $= 0$. If n be odd, therefore, this value of x does not correspond to either a maximum or minimum; and if n be even, it will be found that the value $x = a$ renders the nth differential coefficient > 0, and that therefore the function is a minimum.

Similar observations apply to the value $x = 0$, by considering $x - 0$ as a factor of the differential coefficients.

The value $x = \frac{ma}{m+n}$ being substituted for x in $\frac{d^2u}{dx^2}$, renders it negative, and therefore renders the function a maximum.

Ex. 3. Let $u = \frac{x}{1+x^2}$. In this case let $uu' = 1$; when u' is a maximum, u is a minimum, and *vice versa*. But $u' = x + \frac{1}{x}$, $\because$

$$\frac{du'}{dx} = 1 - \frac{1}{x^2},$$
$$\frac{d^2u'}{dx^2} = \frac{2}{x^3}.$$

The condition $\frac{du'}{dx} = 0$, gives $x = \pm 1$, $\because \frac{d^2u'}{dx^2} = \pm 2$.

Hence if $x = 1$, $\because u' = 2$, a minimum, and $u = \frac{1}{2}$, a maximum. If $x = -1$, $\because u = -2$, a maximum, and $u = -\frac{1}{2}$, a minimum.

The principle used here frequently abridges the process, scil. by investigating the maximum or minimum of the reciprocal of the function in place of the function itself.

(121.) The maxima and minima values of functions of several variables are determined upon principles similar to those which have been already applied to functions of a single variable. If $u = \text{F}(x', x'', x''' \ldots.)$ and $u' = \text{F}(x' \pm h', x'' \pm h'', x''' \pm h''' \ldots.)$; let such a system of values be supposed to be assigned to the variables $x', x'', x''' \ldots.$ as renders the sign of $u - u'$ independent of the signs of the quantities $h', h'', h''' \ldots.$ these quantities having any system of finite values, however small, and such, that the quantity $u - u'$ will preserve the same sign for all systems of values of $h', h'', h''' \ldots.$ between the assumed system and $h' = 0$, $h'' = 0$, $h''' = 0 \ldots.$ the value of u, which corresponds to the system of values of the variables thus assumed, is a maximum if the sign of $u - u'$ be positive, and a minimum if its sign be negative.

(122.) We shall first consider the case where u is a function of two variables. In this case

$$u - u' = \left\{ \mp \frac{du}{dx'} \cdot \frac{h'}{1} \mp \frac{du}{dx''} \cdot \frac{h''}{1} \right\} - \left\{ \frac{d^2u}{dx'^2} \cdot \frac{h'^2}{1.2} \pm \frac{d^2u}{dx'dx''} \cdot \frac{h'h''}{1} + \frac{d^2u}{dx''^2} \cdot \frac{h''^2}{1.2} \right\} + \quad \ldots \ldots \quad [1].$$

In order that the sign of $u - u'$ may be independent of h' and h'', it is necessary that such values be assigned to the variables as will render

$$\frac{du}{dx'} = 0, \; \frac{du}{dx''} = 0 \quad \ldots \ldots \quad [2].$$

These equations will in general give determinate values of x' and x''. In order, however, to find whether this system of values of the variables gives a maximum or minimum value of the function, it will be necessary to substitute them in the differential coefficients of the second order, and to determine whether the sign of the quantity

$$\frac{d^2u}{dx'^2}\cdot\frac{h'^2}{1.2}\pm\frac{d^2u}{dx'dx''}\cdot\frac{h'h''}{1}+\frac{d^2u}{dx''^2}\cdot\frac{h''^2}{1.2}\quad .\ .\ .\ .\ [3],$$

is independent of the signs of h' and h''. For this purpose, let $\frac{h''}{h'} = k$, and the preceding formula becomes

$$\frac{h'^2}{2}\left\{\frac{d^2u}{dx'^2}\pm 2\frac{d^2u}{dx'dx''}.k+\frac{d^2u}{dx''^2}.k^2\right\}.$$

If the sign of this be independent of k, the values of k, which render it $= 0$, must be imaginary. This gives the condition

$$\frac{d^2u}{dx'^2}\cdot\frac{d^2u}{dx''^2}-\left(\frac{d^2u}{dx'dx''}\right)^2>0\quad .\ .\ .\ .\ .\ .\ .\ [4].$$

That this condition may be fulfilled, it is necessary that $\frac{d^2u}{dx'^2}$ and $\frac{d^2u}{dx''^2}$ should have the same sign.

If $h'' = 0$, the quantity [3] becomes $\frac{d^2u}{dx'^2}\frac{h'^2}{1.2}$, and the sign of this quantity must therefore be the sign of [3], since it always retains the same sign. Hence it follows, that

1°. If any system of values of x' and x'', determined by [2], give $\frac{d^2u}{dx'^2}$, $\frac{d^2u}{dx''^2}$ different signs, the function has no corresponding maximum or minimum.

2°. If any such system of values of x' and x'' give $\frac{d^2u}{dx'^2}$, $\frac{d^2u}{dx''^2}$ the same sign, and yet render

$$\frac{d^2u}{dx'^2} \cdot \frac{d^2u}{dx''^2} - \left(\frac{d^2u}{dx'dx''}\right)^2 < 0 \text{ or } = 0,$$

the function has no corresponding maximum or minimum.

3°. If such a system of values of x', x'', give $\frac{d^2u}{dx'^2}$ and $\frac{d^2u}{dx''^2}$, a negative sign, and also fulfil the condition [4], the corresponding value of the function is a maximum.

4°. If such a system render $\frac{d^2u}{dx'^2}$, $\frac{d^2u}{dx''^2}$, both positive, and also fulfil the condition [4], the corresponding value of the function is a minimum.

(123.) It may happen that the system of values of x', x'', determined by [2], also fulfil the conditions

$$\frac{d^2u}{dx'^2} = 0, \ \frac{d^2u}{dx''^2} = 0, \ \frac{d^2u}{dx'dx''} = 0.$$

In this case it will be necessary to substitute them in the partial differential coefficients of the third order.

If they do not render these $= 0$, the function admits of no corresponding maximum or minimum; but if they do, it is necessary to examine the effect of the same substitution on the differential coefficients of the fourth order. The terms of the development involving h', h'', in four dimensions, being treated as those involving two, and the conditions of imaginary roots determined, similar conclusions follow, and so the investigation may be continued as in functions of a single variable.

(124.) Similar reasoning may easily be applied to functions of any number of variables. The conditions which determine the system of values of the variables which *may* give a maximum or minimum, are

$$\frac{du}{dx'} = 0, \ \frac{du}{dx''} = 0, \ \frac{du}{dx'''} = 0, \ \ldots \ [5].$$

But to determine if any and what system of values of the

variables derived from these equations *must* give a maximum or minimum, it will be necessary to examine their effects upon the successive partial differential coefficients.

It frequently happens that some one or more of the equations [5] can be inferred from the others. In this case the number of independent equations being less than the number of quantities to be determined, it follows that there are an infinite number of systems of values which may all determine maxima or minima.

In this case, if the question be geometrical, the solution is a *locus*.

(125.) We shall now give some examples of the investigation of maxima and minima of functions of several variables.

Ex. 1. To divide a quantity a into three parts, x, y, and $a-x-y$, such that the product $u = x^m.y^n.(a-x-y)^p$ is a maximum or minimum.

The differential coefficients of the first order are

$$\frac{du}{dx} = x^{m-1}.y^n.(a-x-y)^{p-1}.\{ma - mx - my - px\},$$

$$\frac{du}{dy} = x^m.y^{n-1}.(a-x-y)^{p-1}.\{na - nx - ny - py\}.$$

The factors within the latter parenthesis of each being put $= 0$, and solved, give

$$x = \frac{ma}{m+n+p}, \quad y = \frac{na}{m+n+p}, \quad a-x-y = \frac{pa}{m+n+p}.$$

In order to discover whether these correspond to a maximum or a minimum, we must substitute them in the general expressions for

$$\frac{d^2u}{dx^2}, \frac{d^2u}{dy^2}, \frac{d^2u}{dxdy}.$$

And if $m+n+p$ be called q, we find

$$\frac{d^2u}{dx^2} = -(m+p)\left(\frac{ma}{q}\right)^{m-1}\left(\frac{na}{q}\right)^n.\left(\frac{pa}{q}\right)^{p-1},$$

$$\frac{d^2u}{dy^2} = -(n+p)\left(\frac{ma}{q}\right)^m \cdot \left(\frac{na}{q}\right)^{n-1} \cdot \left(\frac{pa}{q}\right)^{p-1},$$

$$\frac{d^2u}{dxdy} = -\frac{mna}{q}\left(\frac{ma}{q}\right)^{m-1} \cdot \left(\frac{na}{q}\right)^{n-1} \cdot \left(\frac{pa}{q}\right)^{p-1}.$$

The quantities $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dy^2}$, are both negative, and fulfil the condition [4], and therefore the corresponding value of the function is a maximum.

We shall not pursue here the investigation of the consequences of the other factors of the above equations being $= 0$, as the student can readily do it himself.

Ex. 2. *To find the greatest triangle which can be included within a given perimeter.*

Let the perimeter be $2a$, the sides x, y, and $2a - x - y$, and the area u. By a well known principle,

$$u = \sqrt{a(a-x)(a-y)(x+y-a)}.$$

Assuming the logarithms,

$$2lu = la + l(a-x) + l(a-y) + l(x+y-a).$$

Differentiating for x and y, we find

$$2\frac{du}{u} = -\frac{dx}{a-x} + \frac{dx}{x+y-a},$$

$$\therefore \frac{du}{dx} = \tfrac{1}{2}u\frac{2a-2x-y}{(a-x)(x+y-a)},$$

$$\frac{du}{dy} = \tfrac{1}{2}u.\frac{2a-2y-x}{(a-y)(x+y-a)}.$$

The conditions under which these $= 0$ are

$$2a - 2x - y = 0,$$
$$2a - 2y - x = 0.$$

Hence

$$x = \tfrac{2}{3}a,$$
$$y = \tfrac{2}{3}a,$$
$$2a - x - y = \tfrac{2}{3}a.$$

Hence the triangle is equilateral. It is evident from the nature of the question, that this result determines a maximum. However, this may be proved by examining the partial differential coefficients of the second order by the criterion which has been already established.

Ex. 3. Let

$u^2 = (\text{L} + \text{Y}x - \text{X}y)^2 + (\text{M} + \text{X}z - \text{Z}x)^2 + (\text{N} + \text{Z}y - \text{Y}z)^2.$

This being differentiated for x, y, and z, and the partial differential coefficients being made $= 0$, the results, after reduction, and putting R^2 for $\text{X}^2 + \text{Y}^2 + \text{Z}^2$, are

$$\text{X}(\text{X}x + \text{Y}y + \text{Z}z) + \text{MZ} - \text{LY} - \text{R}^2x = 0,$$
$$\text{Y}(\text{X}x + \text{Y}y + \text{Z}z) + \text{LX} - \text{NZ} - \text{R}^2y = 0,$$
$$\text{Z}(\text{X}x + \text{Y}y + \text{Z}z) + \text{NY} - \text{MX} - \text{R}^2z = 0.$$

If these equations were independent, they would give a determinate system of values of xyz. But they are not independent; for if the first be multiplied by X, the second by Y, and the third by Z, an equation will result independent of xyz, whose terms will destroy one another. If the quantity within the parenthesis be eliminated, the equations will become

$$\text{Y}x - \text{X}y + \text{L} = \frac{\text{Z}(\text{LZ} + \text{MY} + \text{NX})}{\text{R}^2},$$

$$\text{Z}y - \text{Y}z + \text{N} = \frac{\text{X}(\text{LZ} + \text{MY} + \text{NX})}{\text{R}^2},$$

$$\text{X}z - \text{Z}x + \text{M} = \frac{\text{Y}(\text{LZ} + \text{MY} + \text{NX})}{\text{R}^2}.$$

This question comes under the observation in (124.), and it follows, that there are an infinite number of systems of values which determine the maximum or minimum value of the function. If in this case xyz be the co-ordinates of a point in space, the locus of that point is a straight line represented by the above equations, and the value of u for all values of xyz is

$$u = \frac{\text{LZ} + \text{MY} + \text{NX}}{(\text{X}^2 + \text{Y}^2 + \text{Z}^2)^{\frac{1}{2}}}.$$

This question is connected with the theory of statical moments, and the right line thus determined is the locus of the points of minimum principal moment. (See *Poisson, Traité de Mécanique*, livre i. chap. 3).

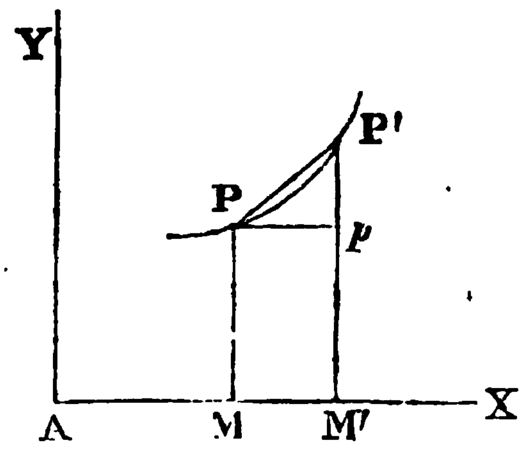
SECTION XII.

Application of the Differential Calculus to the Geometry of Plane Curves. Arcs and Areas. Principles of Contact.

OF ARCS AND AREAS.

PROP. LVI.

(126.) *To determine the differential of the arc of a curve considered as a function of the co-ordinates of its extremities.*



By the equation of the curve, y is a function of x.

Let $\text{AM} = x$, $\text{PM} = y$, $\text{MM}' = h$, $\text{P}'\text{M}' = y' = \text{F}(x+h)$. By Taylor's series,

$$\text{P}'\text{M}' - \text{PM} = \frac{dy}{dx} \cdot \frac{h}{1} + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} + \cdots$$

The co-ordinates being rectangular, $\text{PP}' = \sqrt{h^2 + (\text{P}'p)^2}$, ∴ the value of PP' must have the form

$$\text{PP}' = \sqrt{h^2 + \frac{dy^2}{dx^2} \cdot \frac{h^2}{1} + \text{A}h^3 + \text{B}h^4 \cdots}$$

The limit of the ratio of the arc PP$' = s$, and its chord being a ratio of equality, it is evident that when $h = 0$, $\frac{PP'}{h} = \frac{ds}{dx}$.

But

$$\frac{PP'}{h} = \sqrt{1 + \frac{dy^2}{dx^2} + Ah + Bh^2 + \cdots}$$

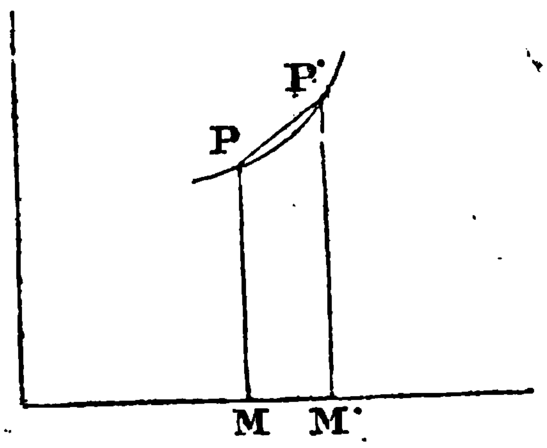
When $h = 0$, this becomes

$$\frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}},$$

$$\therefore ds = \sqrt{dy^2 + dx^2}.$$

PROP. LVII.

(127.) *To express the differential of the area included by a curve, and the ordinates of any two points upon it, as a function of the co-ordinates.*



Since the arc and chord PP' coincide when h or MM' is indefinitely diminished, the limit of the ratio of the area included by the ordinates and the arc PP', to that included by the ordinates and the chord PP', is a ratio of equality. Let da be the differential of the area. It is evident that when $h = 0$,

$$\frac{PP'M'M}{h} = \frac{da}{dx}.$$

But

$$PP'M'M = MM' \times \tfrac{1}{2}(PM + P'M') = \tfrac{1}{2}h(y + y'),$$

$$y' = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2}, \cdots$$

$$\therefore \tfrac{1}{2}(y' + y) = y + \frac{dy}{dx}\cdot\frac{h}{1.2} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2^2}, \cdots$$

$$\therefore \tfrac{1}{2}h(y' + y) = yh + \frac{dy}{dx}\cdot\frac{h^2}{1.2} + \frac{d^2y}{dx^2}\cdot\frac{h^3}{1.2^2}\cdots$$

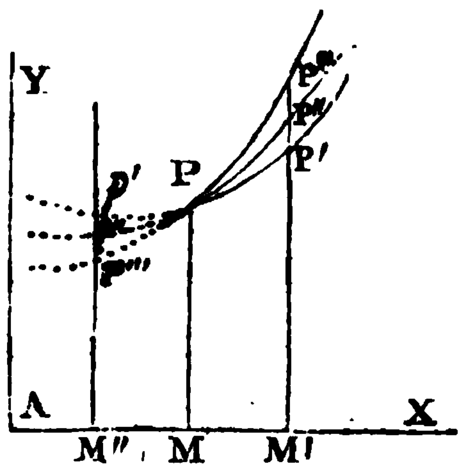
$$\therefore \frac{\text{PP}'\text{M}'\text{M}}{h} = y + \frac{dy}{dx}\cdot\frac{h}{1.2} + \frac{d^2y}{dx^2}\cdot\frac{h^3}{1.2^3}\cdots$$

When $h = 0$ therefore we obtain

$$\frac{da}{dx} = y, \therefore da = ydx.$$

OF CONTACT.

(128.) Let the equations of threeplane curves passing through the same point P be $\text{F}(xy)=0$, $\text{F}'(xy)=0$, $\text{F}''(xy)=0$, and let these equations be related to the same axes of co-ordinates AY, AX. Let the co-ordinates of the common point P be yx, and let y', y'', y''', be what y becomes in each of the equations when x becomes $x + h$. Let $\text{MM}' = h$, and $\text{P}'\text{M}' = y'$, $\text{P}''\text{M}' = y''$, $\text{P}'''\text{M}' = y'''$. These values being severally expressed by Taylor's series, are



$$y' = y + \text{A}_1\cdot\frac{h}{1} + \text{A}_2\cdot\frac{h^2}{1.2} + \text{A}_3\cdot\frac{h^3}{1.2.3}\cdots$$

$$y'' = y + \text{B}_1\cdot\frac{h}{1} + \text{B}_2\cdot\frac{h^2}{1.2} + \text{B}_3\cdot\frac{h^3}{1.2.3}\cdots$$

$$y''' = y + \text{C}_1\cdot\frac{h}{1} + \text{C}_2\cdot\frac{h^2}{1.2} + \text{C}_3\cdot\frac{h^3}{1.2.3}\cdots$$

Where $\text{A}_1, \text{A}_2, \cdots \text{B}_1, \text{B}_2, \cdots \text{C}_1, \text{C}_2, \cdots$ express the successive differential coefficients.

Since the first terms of the three series are the same, MM' may be assumed so small, that the order of the magnitudes of the three ordinates y', y'', y''', will be that of the three coefficients A_1, B_1, C_1, if these three be supposed unequal (92.).

Thus the figure is represented as if $A_{\prime}$ were the least, and $C_{\prime}$ the greatest.

If the same negative value MM'' be assigned to h, then the order of the magnitudes of y', y'', y''', must be the opposite of that of $A_{\prime}$, $B_{\prime}$, $C_{\prime}$, therefore, of the three points P', P'', P''', where the curves meet the parallel to the axes of y through M'', P''' is the lowest, and P' the highest. This being opposite to their order on the other side of the point P, it is plain that the curves cross each other at the point P.

(129.) It therefore follows, that the position of the curves in the immediate vicinity of their common point P is to be determined by the relation between the magnitudes of the first differential coefficients.

If two ($A_{\prime}$ and $B_{\prime}$) of the three coefficients be rendered equal by the co-ordinates of the point P, then the relative magnitudes of the ordinates y' and y'' are to be determined by A_2 and B_2, and by assuming h or MM' sufficiently small, y' will be greater or less than y'', according as A_2 is $>$ or $<$ B_2. In this case, also, y''' is at the same time greater or less than both y' and y'', according as $C_{\prime}$ is greater or less than the common value of $A_{\prime}$ and $B_{\prime}$. These conclusions are evident from (92.).

Hence, it follows that if $C_{\prime}$ have not the common value of $A_{\prime}$ and $B_{\prime}$, the curve PP''' cannot pass between the curves PP' and PP'', but must pass either above both or below both, according as $C_{\prime}$ is $>$ or $<$ the common value of $A_{\prime}$ and $B_{\prime}$.

The curves PP' and PP'' in the vicinity of the point P therefore approach each other more closely than the curve PP''' can to either of them. These curves are said in this case to have *contact of the first degree*.

(130.) Let us now suppose that the point P is such that its co-ordinates render the three coefficients $A_{\prime}$, $B_{\prime}$, and $C_{\prime}$, equal. Then by diminishing MM' or h sufficiently, the order of the magnitudes of y', y'', y''', will be determined by that

of the coefficients A_2, B_2, C_2, and the three curves will have contact of the first degree. In this case the change in the sign of h not affecting its square, produces no effect upon the order of the magnitudes of y', y'', y'''; therefore the points P', P'', P''', are in the same order on both sides of the point P, and therefore the curves do not cross each other at that point.

If the co-ordinates of P render $A_2 = B_2$, the order of the magnitudes of y', y'', must be determined by that of A_3, B_3. In this case y''' must be greater or less than both y' and y'', according as C_2 is greater or less than the common value of A_2 and B_2. Hence as before, it follows that no curve for which C_2 is not equal to the common value of A_2 and B_2 can pass between the curves PP' and PP'' in the immediate vicinity of the point P. The two curves PP' and PP'' are in this case said to have *contact of the second degree.*

If the co-ordinates of the point P render the three quantities A_2, B_2, C_2, equal, then the three curves have contact of the second degree. In this case, as the sign of the third term of the series changes with that of h, since it involves h^3, the order of the magnitudes of y', y'', y''', for $+h$ and $-h$ are opposite, and therefore the points $P'P''P'''$ on different sides of the point are in opposite orders. Hence the three curves cross each other at the point P. Thus contact of the second degree is both contact and intersection.

(131.) By pursuing this reasoning, we may conclude in general, that if the co-ordinates of the point P render the successive differential coefficients from the first to the nth inclusive, equal, each to each, no curve which agrees with these in a less number of differential coefficients can pass between them. The two curves are said in this case to have contact of the nth degree. If the contact be of an even degree, the first terms of the two series, which do not agree, involve an odd power of h, the sign of which changes with that of h,

and, therefore, contact of an even degree is both contact and intersection; but if the contact be of an odd degree, the first unequal terms involve an even power of h, of which the sign does not change with that of h, and, therefore, contact of an odd degree is contact without intersection.

(132.) If the equations $F'(xy) = 0$ and $F''(xy) = 0$ be those of right lines, being equations of the first degree with respect to the variables, all the differential coefficients after the first are $= 0$; therefore the series end at the second terms. It follows from what has been already proved, that if $A_{\prime} = B_{\prime\prime}$ and $C_{\prime}$ be not equal to $A_{\prime\prime}$ that the second right line cannot pass between the curve and the first, and if $C_{\prime}$ becomes equal to the common value of $A_{\prime}$ and $B_{\prime\prime}$ the two right lines become identical, since the two series end at these terms. Substituting $x' - x$ for h, it appears that the right line represented by the equation (Geom. (26.)),

$$(y - y') - \frac{dy}{dx}(x - x') = 0$$

meets the curve at P' in such a manner, that no other right line passing through the point P can pass between it and the curve. This right line is therefore a tangent to the curve at the point.

If the co-ordinates be rectangular, $\frac{dy}{dx}$ is the tangent of the angle under the tangent line and the axis of x. Geom. (15.)

If $\frac{dy}{dx} = 0$; the tangent will be parallel to the axis of x; and if $\frac{dy}{dx}$ be infinite, the tangent is parallel to the axis of y.

For the values of the subtangent, and subnormal, and the equation of the normal, see Geom. (323.), *et seq*.

(133.) For a curve, whose equation is of the form,

$$y = a + bx + cx^2.$$

The series for y' terminates at the third term, and

$$\text{B}_{,} = b + 2cx,$$

$$\text{B}_{2} = 2c.$$

If in such a curve the coefficients $\text{B}_{,}$ and B_2 be equal to those of the curve $\text{F}(xy) = 0$, it will touch this curve with contact of the second degree, while no curve of the same kind, that is, whose equation is of the same form, can touch the curve $\text{F}(xy)$ with so intimate a contact. The curve is in this case said to *osculate*. The nature and principles of osculation are so fully explained in my Geometry (353.), that it would be needless repetition to enter upon the subject here.

(134.) The curve represented by the above equation is a parabola (Geom. Sect. VII.). By analogy to this, a class of curves represented by equations of the form

$$y = a + bx + cx^2 + dx^3,$$

. . . .

$$y = a + bx + cx^2 \ldots . gx^m,$$

are called *parabolic* curves, and the series for y' for each of them terminates at the $(m + 1)$th term, all the differential coefficients after the mth being $= 0$. When such curves have a common point P with any proposed curve, and all the terms of the expansions of y' agree with the corresponding terms in the expansion of y' for the proposed curve, they are called *osculating parabolas*. In this sense the osculating parabola of the first order is the rectilinear tangent. The osculating parabola of the second order is the common parabola. The osculating parabola of the third order is the cubical parabola, and so on. It follows, also, from what has been said (131.), that osculating parabolas of even orders both touch and intersect, while those of odd orders touch without intersecting.

(135.) The osculating parabolas furnish means of representing geometrically the successive terms of Taylor's series, or the differential coefficients.

Let $y_{\prime}$, $y_{\prime\prime}$, $y_{\prime\prime\prime}$, &c. be the ordinates of the several osculating parabolas corresponding to $x + h$, so that

$$y_{\prime} = y + \frac{dy}{dx}\cdot\frac{h}{1},$$

$$y_{\prime\prime} = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2},$$

$$y_{\prime\prime\prime} = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3}.$$

. . . .

. . . .

$$\because y_{\prime} - y = \frac{dy}{dx}\cdot\frac{h}{1},$$

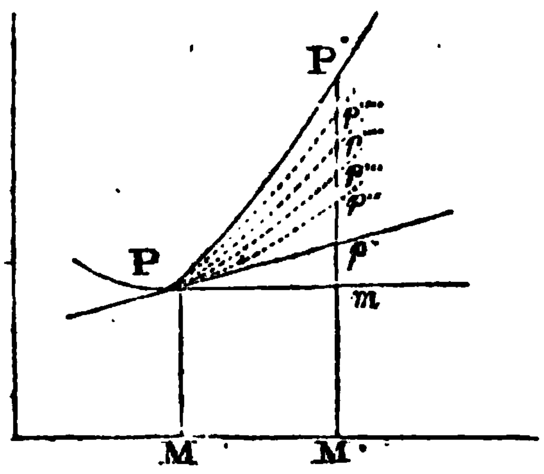
$$y_{\prime\prime} - y_{\prime} = \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2},$$

$$y_{\prime\prime\prime} - y_{\prime\prime} = \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3}.$$

. . . .

. . . .

$$y_{(n)} - y_{(n-1)} = \frac{d^ny}{dx^n}\cdot\frac{h^n}{1.2.3.\ldots.n}.$$



Let $MM' = h$, and Pp' being the tangent, let Pp'', Pp''', Pp'''', &c. be the successive osculating parabolas, then $M'm$, mp', $p'p''$, $p''p'''$, $p'''p''''$, &c. are the successive terms of Taylor's series; and if dx be assumed $= h$, then

$$M'm = y, \quad mp' = dy,$$

$$1.2(p'p'') = d^2y, \quad 1.2.3(p''p''') = d^3y,$$

. . . .

. . . .

(136.) The order of osculation of a curve of any proposed degree depends on the number of constants which enter its equation (Geometry, 353). The curve of the second degree, which osculates any proposed curve, touches it therefore with contact of the fourth order, and the coefficients of the equation of this osculating curve are functions of the constants in the equation of the proposed curve and the co-ordinates of the point of contact. Let the equation of the osculating curve be

$$Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0;$$

the species of this curve is to be determined by the quantity $B^2 - 4AC$, which being a function of the co-ordinates of the point of contact, varies from point to point of the proposed curve. Suppose y eliminated by means of the equation of the curve, then $B^2 - 4AC$ becomes a function of x alone. Let the roots of the equation

$$B^2 - 4AC = 0$$

be x', x'', x''' .... At the points of the curve, which correspond to the real roots of this equation, the osculating curve is a parabola. And since $B^2 - 4AC$ changes its sign in passing through 0, it follows that the osculating curve on one side of such a point is an ellipse, and on the other side an hyperbola; the species changing as often as there are real values of y corresponding to the real root of the above equation.

If the roots of the equation be all imaginary, the quantity $B^2 - 4AC$ always retains the same sign, and therefore the osculating curve always remains of the same species.

If the condition $B^2 - 4AC = 0$ be fulfilled independently of xy by the constants of the given equation, then the osculating curve for all points is a parabola.

Similar observations may be applied to osculating curves of any proposed degree.

Although the degree of contact of an osculating curve of

any species depends on the number of constants which enter its equation, yet it may happen at *particular* points of the given curve, that the contact is of a higher degree than that which marks in general the order of its osculation. This circumstance arises from an additional differential coefficient of the given curve being rendered equal to the corresponding differential coefficient of the curve which osculates it, by the peculiar values of the co-ordinates of the point of contact. We shall soon meet an example of this.

SECTION XIII.

Of osculating circles and evolutes.

(137.) The most remarkable osculating curve is the circle.

The equation of the circle, involving three constant quantities, the order of its osculation is the second.

Let

$$(y-y')^2+(x-x')^2=\text{R}^2$$

be the equation of a circle, whose radius is R, and the co-ordinates of whose centre are $x'y'$. The first and second differential coefficients are

$$\text{B}_1=-\frac{x-x'}{y-y'},$$

$$\text{B}_2=-\frac{\text{R}^2}{(y-y')^3},$$

yx being the co-ordinates of the point of contact, it is necessary that the quantities x', y', and R, should receive such values (130.), that

$$\text{A}_1=\text{B}_1,\quad \text{A}_2=\text{B}_2.$$

To determine the values of y', x', and R, which fulfil these conditions, let the values of $B_{,}$ and $B_{,,}$ already found, be substituted for them, and the equations

$$(y-y')^2+(x-x')^2=R^2,$$
$$A_{,}(y-y')+(x-x')=0,$$
$$A_{2}(y-y')^3+R^2=0,$$

give

$$y'=y+\frac{1+A_{,}^2}{A_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [1],$$

$$x'=x-\frac{1+A_{,}^2}{A_2}A_{,} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [2],$$

$$R^2=-A_2\left(\frac{1+A_{,}^2}{A_2}\right)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [3],$$

or substituting for $A_{,}$ and A_2 their values

$$y'=y+\frac{dy^2+dx^2}{d^2y},$$

$$x'=x-\frac{dy^2+dx^2}{d^2y}\cdot\frac{dy}{dx},$$

$$R=-\frac{(dy^2+dx^2)^{\frac{3}{2}}}{d^2y.dx}.$$

(138.) The equality $A_{,}=B_{,,}$ which gives

$$\frac{dy}{dx}(y-y')+(x-x')=0,$$

shows that the centre of a circle having a common rectilinear tangent with the curve, must be upon the normal (Geom. 325).

The radius of the osculating circle is generally called the *radius of curvature*. (Geom. 335).

(139.) Since y, $A_{,}$, and A_2, are functions of x by the equation of the given curve and its differentials, it is evident that y' and x' are implicit functions of x. If, therefore, x varies by assuming the values corresponding to the different points of the curve, the quantities $y'x'$ suffer consequent variations, and the centre of the osculating circle

assumes different positions accordingly. The locus of this centre is called the *evolute* of the curve, and its equation may be found by eliminating x by the equations which give $x'y'$ as functions of x. (Geom. 337, 338, and notes).

(140.) Since $ds^2 = dy^2 + dx^2$ (126), ds being the differential of the arc of the curve, $\therefore$

$$\text{R} = -\frac{ds^3}{d^2y dx}.$$

The expressions already found for the radius of curvature are determined on the supposition that x is the independent variable. To obtain its value, independently of this hypothesis, it is only necessary to substitute for $\frac{d^2y}{dx^2}$ (38),

$$\frac{dxd^2y - dyd^2x}{dx^3},$$

which will give

$$\text{R} = -\frac{(dy^2+dx^2)^{\frac{3}{2}}}{dxd^2y - dyd^2x},$$

$$\text{or R} = -\frac{ds^3}{dxd^2y - dyd^2x}.$$

If s be considered as the independent variable $d(ds^2) = 0$,

$$\therefore dyd^2y + dxd^2x = 0.$$

Squaring this, and adding it to the denominator of the above value squared, we find

$$\text{R}^2 = \frac{ds^6}{[(d^2y)^2 + (d^2x)^2](dy^2 + dx^2)};$$

or since $ds^2 = dy^2 + dx^2$, $\therefore$

$$\text{R} = \frac{ds^2}{\sqrt{(d^2y)^2 + (d^2x)^2}}.$$

This expression is used by Laplace. See *Mecanique Celeste*, liv. i. chap. 2.

Another expression, frequently used by physical authors for the radius of curvature, is

$$\text{R} = \frac{ds}{d\varphi},$$

$d\varphi$ being the angle under the normals through the extremities of ds. As ds may be considered coincident with the arc of the osculating circle, it is evident that since $d\varphi$ is the angle under the two radii through the extremities of ds, we have $\text{R}d\varphi = ds$.

PROP. LVIII.

(141.) *To determine the position of the tangent to the evolute at any point* y'x' *corresponding to* xy *upon the given curve.*

Let the values $y'x'$, already found, be differentiated as functions of x, and the results are

$$dy' = dy + d\cdot\frac{1+\text{A}_{\prime}^{2}}{\text{A}_{2}},$$

$$dx' = dx - (1 + \text{A}_{\prime}^{2})dx - \text{A}_{\prime}d\cdot\frac{1+\text{A}_{\prime}^{2}}{\text{A}_{2}}.$$

Multiplying the first by $\text{A}_{\prime}$, and adding

$$\text{A}_{\prime}dy' + dx' = \text{A}_{\prime}dy - \text{A}_{\prime}^{2}dx;$$

but $dy = \text{A}_{\prime}dx$, $\because$

$$\text{A}_{\prime}dy' + dx' = 0,$$

$$\therefore \frac{dy'}{dx'} = -\frac{1}{\text{A}_{\prime}} = -\frac{dx}{dy}.$$

Hence the equation of the tangent at the point $y'x'$ is

$$(y - y')dy + (x - x')dx = 0.$$

Hence the normal to the curve is tangent to the evolute. (Geom. 341).

PROP. LIX.

(142.) *To determine the change in the arc of the evolute corresponding to any proposed change in the radius of curvature.*

In the equation

$$R^2 = (y - y')^2 + (x - x')^2,$$

y, y', and x', being functions of x already determined, R may be differentiated as a function of x, $\therefore$

$$R\,dR = (y - y')(dy - dy') + (x - x')(dx - dx'),$$

$$\therefore \frac{R\,dR}{y-y'} = dy - dy' + \frac{x-x'}{y-y'}(dx - dx').$$

Substituting for $\frac{x-x'}{y-y'}$, its value $-\frac{dy}{dx}$, the result is

$$\frac{R}{y-y'}\,dR = -dy' + \frac{dy}{dx}\cdot dx';$$

but $\frac{dy}{dx} = -\frac{dx'}{dy'}$. Making this substitution, and squaring, we find

$$\left\{1 + \frac{(x-x')^2}{(y-y')^2}\right\} dR^2 = \frac{(dy'^2 + dx'^2)^2}{dy'^2},$$

$$\text{or } dR^2 = dy'^2 + dx'^2,$$

$$\therefore dR = (dy'^2 + dx'^2)^{\frac{1}{2}},$$

observing that

$$1 + \frac{(x-x')^2}{(y-y')^2} = 1 + \frac{dy^2}{dx^2} = 1 + \frac{dx'^2}{dy'^2}.$$

Hence it follows, that the increment of the arc of the evolute is equal to the simultaneous increment of the radius of curvature, and the property from whence the evolute has derived its name may be thence deduced. (Geometry, 342).

PROP. LX.

(143.) *To determine the point upon any curve at which the radius of curvature is a maximum or minimum.*

By (137.),

$$R = -\frac{ds^3}{d^2ydx}.$$

Differentiating and equating the result with zero, we find

$$(y' - y)d^3y - 3dyd^2y = 0.$$

Substituting in this equation for y', y, and the differentials, their values as functions of x, the roots will determine the sought points.

PROP. LXI.

(144.) *To determine the points of a curve at which the contact of the osculating circle is of the third degree.*

The third differential coefficient derived from the equation of the osculating circle is

$$B_3 = -\frac{3dyd^2y}{(y-y')dx^3}.$$

Equating this with $\frac{d^3y}{dx^3}$ derived from the curve, the result is

$$(y' - y)d^3y - 3dyd^2y = 0.$$

This equation being identical with that found in the last proposition, it follows that the contact is of the third order at the points of greatest and least curvature.

SECTION XIV.

Of asymptotes.

(145.) Let the equations of two plane curves which have infinite branches be

$$F(xy) = 0, \quad F'(xy') = 0,$$

y and y' being the values of y in the two curves corresponding to the same value of x. The distance between the curves measured in a direction parallel to the axis of y is $y - y'$. If, as x increases without limit, either positively or negatively, the distance $y - y'$ diminishes without limit, but vanishes only when x becomes infinite, the infinite branch of the one curve is said to be an asymptote to the other. (Geom. 345).

In order that this should occur, it is necessary that the quantity $y - y'$, developed according to the powers of x, should contain only negative powers of x. For if it contained a positive power, $y - y'$ would be rendered infinite by x becoming infinite, and if it contained a term independent of x, it would be finite when x is infinite.

Hence the development of $y - y'$ must have the form

$$y - y' = Ax^{-a} + Bx^{-b} + \cdots$$

the exponents being supposed to descend.

It follows, therefore, that if the development of y by the descending powers of x contain any positive powers or a term independent of x, all these must also occur in the development of y', in order that they may disappear by subtraction. Hence, if the development of y be

$$y = A'x^{a'} + B'x^{b'} \cdots M + Ax^{-a} + Bx^{-b} \cdots$$

the development of y' must be

$$y' = A'x^{a'} + B'x^{b'} \cdots\cdots M + \cdots\cdots$$

the terms which succeed M, or those which involve negative powers of x, being unrestricted.

(146.) Since the terms of the development which succeed M are arbitrary, it follows that there may be an infinite number of asymptotes to the same curve, and that each of these will be asymptotes to each other. The most simple asymptote which the curve admits, at least that whose development is simplest, is the curve represented by the equation

$$y' = A'x^{a'} + B'x^{b'} + \cdots\cdots M.$$

The curve represented by

$$y'' = A'x^{a'} + B'x^{b'} + \cdots\cdots M + Ax^{-a}$$

is also an asymptote, and approaches closer to the curve than the former, since, by increasing x, it is manifest that y'' approaches nearer to equality with y than y' does.

In like manner, the curve represented by

$$y''' = A'x^{a'} + B'x^{b'} + \cdots\cdots M + Ax^{-a} + Bx^{-b}$$

has asymptotism of a still higher order with the given curve.

(147.) Thus it appears that there are orders of asymptotism in some degree analogous to the orders of contact. Curves which admit asymptotes are sometimes divided into *hyperbolic* and *parabolic*. Hyperbolic are those which admit a rectilinear asymptote; *parabolic* those which do not.

All hyperbolic curves must therefore be involved in the class

$$y = A'x + B' + Ax^{-a} + Bx^{-b} \cdots\cdots$$

The equation of the rectilinear asymptote being

$$y' = A'x + B'.$$

If $a = 1$ and B, &c. $= 0$, this curve is the common hyperbola.

If $A' = 0$, the asymptote is parallel to the axis of x, and if $A' = 0$, $B' = 0$, the asymptote is the axis of x itself.

(148.) We shall now give some examples illustrative of the preceding theory.

Ex. 1. Let $y = \pm \frac{b}{a}(x^2 - a^2)^{\frac{1}{2}}$. Expanding by the binomial theorem

$$y = \pm \frac{b}{a}x \mp \tfrac{1}{2}bax^{-1} \ldots\ldots$$

Hence the curve has two rectilinear asymptotes represented by the equations

$$y' = \pm \frac{b}{a}x.$$

See Geometry (232.).

Ex. 2. Let $yx = c^2$, $\because$

$$y = c^2x^{-1},$$
$$x = c^2y^{-1}.$$

Hence the asymptotes are the axes of co-ordinates themselves.

Ex. 3. $y^2(x^2 - a^2) = b^4$. By developing, we find

$$y = \pm b^2x^{-1} + \ldots\ldots$$
$$x = \pm a \pm \tfrac{1}{2}\frac{b^4}{a}y^{-2} + \ldots\ldots$$

Hence the axis of x is an asymptote, and there are two other rectilinear asymptotes parallel to the axis of y represented by the equations

$$x = \pm a.$$

There are also two hyperbolæ, $yx = b^2$, and $yx = -b^2$, which are asymptotes.

Ex. 4. Let $y^3 - 3axy + x^3 = 0$, $\because$

$$y = -x - a - a^2x^{-1} - \ldots\ldots$$

There is a rectilinear asymptote represented by

$$y = -a - x.$$

Ex. 5. $y^2x - px^2 - a^3 = 0$. Hence

$$y^2 = px + a^3x^{-1}.$$

Therefore the asymptote to this curve is a common parabola.

(149.) There is, however, another method of determining whether a curve admits a rectilinear asymptote, which is frequently more easily applied than the general method already given. Let the equation of the tangent through any point $y'x'$ on the curve be

$$(y-y')-\frac{dy'}{dx'}(x-x')=0.$$

Let X be the intercept of the axis of x between the origin and the point where the tangent meets the axis of x; and let Y be the corresponding intercept on the axis of y. It is evident that these are obtained by supposing y and x successively $=0$ in the equation of the tangent.

Hence we find

$$\text{X}=\frac{x'dy'-y'dx'}{dy'},$$

$$\text{Y}=\frac{y'dx'-x'dy'}{dx'}.$$

In these quantities let x' be supposed to be increased without limit. If the limits of X and Y be finite, they will determine a rectilinear asymptote.

If X have a limit, but Y none, then the asymptote is parallel to the axis of y at the distance X.

If Y have a limit, but X none, then the asymptote is parallel to the axis of x at the distance Y.

If neither have a limit, or if their values are rendered impossible by increasing x, then the curve has no rectilinear asymptote.

If in the limit $\text{X}=0$ and $\text{Y}=0$ the asymptote passes through the origin, and its direction is found by determining the value of $\frac{dy}{dx}$ when x is indefinitely increased. (See Geom. 346).

These conclusions are founded upon the principle, that

the tangent becomes an asymptote when the point of contact is indefinitely removed.

(150.) Ex. 1. Let $yx + by + cx = 0$. By differentiating

$$\frac{dy}{dx}(x + b) + y + c = 0, \because \frac{dy}{dx} = -\frac{y+c}{x+b}.$$

Hence

$$\text{X} = x' + \frac{y'(x'+b)}{y'+c} = \frac{2y'x' + by' + cx'}{y'+c},$$

$$\text{Y} = y' + \frac{x'(y'+c)}{x'+b} = \frac{2y'x' + by' + cx'}{x'+b},$$

$$\because \text{X} = \frac{y'x'}{y'+c}, \text{Y} = \frac{y'x'}{x'+b}.$$

By solving the equation for y, we find

$$y = -\frac{cx}{b+x} = -\frac{c}{\frac{b}{x}+1}.$$

Hence, when x is infinite, $y = -c$. Also,

$$\text{Y} = \frac{y'}{1+\frac{b}{x'}},$$

which, when x becomes infinite, gives

$$\text{Y} = -c.$$

In a similar way, we find

$$\text{X} = -b.$$

Hence there are two asymptotes parallel to the axes of coordinates. Geom. (123.).

SECTION XV.

Of the direction of curvature—Of the singular points at which a differential coefficient assumes the form $\frac{0}{0}$.

(151.) The development of the value of y corresponding to $x + h$ by Taylor's series, conducts us to a method of determining the direction of the curvature of a curve. Let y'' be the value of y in the equation of the tangent corresponding to $x + h$. Hence

$$y' = y + \frac{dy}{dx} \cdot \frac{h}{1} + \frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} + \frac{d^3y}{dx^3} \cdot \frac{h^3}{1.2.3} \cdots$$

$$y'' = y + \frac{dy}{dx} \cdot \frac{h}{1},$$

$$\therefore y' - y'' = \frac{d^2y}{dx^2} \cdot \frac{h^2}{1.2} + \frac{d^3y}{dx^3} \cdot \frac{h^3}{1.2.3} \cdots$$

Hence $y' - y''$ has the same sign with $\frac{d^2y}{dx^2}$.

Therefore, if y' and y'' be $+$, $y' > y''$ when $\frac{d^2y}{dx^2} > 0$, and $< y''$ when $\frac{d^2y}{dx^2} < 0$, $\therefore$ if $\frac{d^2y}{dx^2} > 0$, the curve is convex towards the axis of x, and if $\frac{d^2y}{dx^2} < 0$, it is concave towards the axis of x. In like manner, if y' and y'' be negative, it is convex or concave towards the axis of x, according as $\frac{d^2y}{dx^2} < 0$ or > 0.

In general, therefore, if y' and $\frac{d^2y}{dx^2}$ have the same sign, the curve is convex toward the axis of x; and if they have different signs, it is concave in that direction.

(152.) We shall now consider the effect produced upon the curve by the differential coefficients becoming $= 0$, or $= \frac{0}{0}$.

If the first differential coefficient be $= 0$, it has been already shown that the tangent is parallel to the axis of x. Such points are therefore thus determined. Let the differential of the proposed equation be

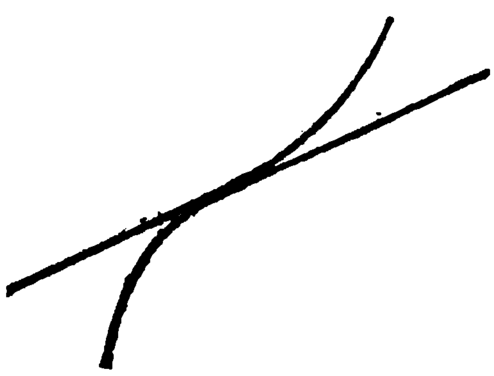
$$\text{A}p + \text{B} = 0,$$

p being the first differential coefficient. Let the values of x and y which satisfy the equations

$$\text{B} = 0, \quad \text{F}(xy) = 0,$$

be determined, and let such systems of values be selected as do not also satisfy $\text{A} = 0$. Such systems of values, if real, determine the points of the curve where the tangent is parallel to the axis of x.

(153.) If the second differential coefficient $= 0$. Since in the equation of the tangent the second differential coefficient also $= 0$, the tangent must have contact of the second degree with the curve. Now, since contact of the second degree is accompanied by intersection (131.), it follows that, at the point thus determined, the curve passes from one side of its tangent to the other, as in the annexed figure.



Such a point of a curve is called *a point of inflexion*, and sometimes *a point of contrary flexure*.

At such a point it is evident that the radius of curvature becomes infinite, since the second differential coefficient is a factor of its denominator (137.).

(154.) If the third differential coefficient be $= 0$, the curve at the corresponding point has contact of the third order with the osculating parabola (134.) of the second order.

And in like manner, if the nth differential coefficient $= 0$, the curve at the corresponding point has contact of the nth degree with the $(n - 1)$th osculating parabola.

(155.) If several successive differential coefficients from the nth to the $(n + p)$th inclusive $= 0$, the curve has contact of the $(n + p)$th order with its osculating parabola of the $(n - 1)$th order.

The effect, therefore, of any combinations whatever of the differential coefficients becoming $= 0$ will be easily perceived.

(156.) Let us next examine the curve at those points where a differential coefficient assumes the form $\frac{0}{0}$.

If the first differential equation be

$$\text{A}p + \text{B} = 0,$$

let systems of values of the variables xy be selected, which at the same time fulfil the equations

$$\text{A} = 0, \quad \text{B} = 0,$$
$$\text{F}(xy) = 0.$$

Such values render the first differential coefficient $= \frac{0}{0}$.

In this case, in order to determine the true value of p, it will be necessary to proceed to the second differential equation (111.), which will give an equation of the form

$$\text{A}'p^2 + \text{B}'p + \text{C}' = 0$$

to determine p.

If this equation be not fulfilled by its coefficients, its roots must either be real and unequal, real and equal, imaginary or infinite.

First. If they be real and unequal, there being two unequal values of the first differential coefficient corresponding to the same values of x and y, there will be consequently two tangents to the curve at the corresponding point; therefore two branches must intersect at that point. Such a point is called a *double point.*

Secondly. If the roots be real and equal, there is but one

value of the differential coefficient, and this presents no particular circumstance in the course of the curve.

Thirdly. If the roots be imaginary, the development representing y' becomes imaginary for both $+h$ and $-h$, and therefore the point whose co-ordinates produce this effect stands alone, insulated, and not continuously connected with any part of the curve. Such is called a *conjugate point.*

The case where a root is infinite will be investigated in the next section.

If, however, the equation of the second degree for p be fulfilled by its coefficients, it will be necessary (111.) to proceed one step further in the differentiation, which will give for the determination of p an equation of the form

$$A''p^3 + B''p^2 + C''p + D'' = 0.$$

If the roots of this equation be real and unequal, there will be three tangents at the corresponding point, and therefore three branches of the curve will intersect at it. Such is called a *triple point.*

If two of the roots be real and equal, there will be but two values of p, which will give a *double point.* If two be imaginary, or all be equal, there will be but one real value of p; in which case the course of the curve will be marked by no peculiarity.

If, however, this equation also be fulfilled by its coefficients proceeding to a fourth differentiation, we shall find an equation of the fourth degree to determine p. Its roots, if real and unequal, determine a quadruple point; if all imaginary, a conjugate point; and, in general, as many as are real and unequal, determine so many tangents to branches of the curve which intersect at the corresponding point.

It will be necessary, therefore, to continue the differentiation until some equation is found, which, not being satisfied by its coefficients, will give determinate values of p. If it have n real and unequal roots, it will determine a *mul-*

tiple point, at which n branches of the curve intersect. If it have but one real root, no peculiarity marks the curve at the corresponding point. If all its roots be imaginary, the point is a *conjugate point*.

(157.) Let us next suppose that the co-ordinates of the point render the second differential coefficient $= \frac{0}{0}$. In this case its value or values may be determined like those of the first (156.).

First. If it have several unequal real values, there will be as many values for the third term of the development of y', and therefore as many different values of y', and therefore as many different branches of the curve passing through the corresponding point. Since, however, the several values of y' agree as to the second term of the developments, they will all have a common tangent. Such a point comes under the class of *multiple points*, and is characterised by the number of branches which, thus meeting, touch with contact of the first degree. This particular species of multiple point may be called a *point of osculation* *.

Secondly. If the coefficient is found to have but one real value, the corresponding point has no particular character.

Thirdly. If all its values be imaginary, it is a conjugate point.

Similar conclusions may be applied to the succeeding differential coefficients, observing that the contact of the branches, which form the point of osculation, is of the $(n-1)$th order, if it be the nth differential coefficient which has the several real values.

(158.) In general, therefore, we find, that in order to determine whether a curve admits a multiple point at which its branches intersect, it will be necessary, 1°. To

* Some French authors call it *un embrassement*.

find the values of xy, which satisfy the equations $\text{A} = 0$, $\text{B} = 0$, $\text{F}(xy) = 0$. 2°. To determine the corresponding values of p. There will be as many intersecting branches as p has real values. If p have no real values, the point is a conjugate point.

In order to determine whether there be a point of osculation, it will be necessary to apply a similar investigation to the superior differential coefficients.

It is obvious from what has been already proved (131.), that at a point of osculation produced by multiple values of a differential coefficient of an odd order, the branches intersect as well as touch; but at one produced by a differential coefficient of an even order, they touch without intersection.

It may happen that the value of p in any of these cases may be infinite.

We shall consider the consequences of this in the next section.

(159.) Ex. 1. To determine whether the curve represented by the equation $ay^3 - x^3y - bx^3 = 0$ has a multiple point.

By differentiating

$$(3ay^2 - x^3)p - 3x^2(y + b) = 0.$$

Hence $\text{A} = 3ay^2 - x^3$, $\text{B} = -3x^2(y+b)$. The only values of xy which render $\text{A} = 0$, $\text{B} = 0$, and also satisfy the equation of the curve are $x = 0$, $y = 0$. To determine the value of p, let the differentiation be continued, and we find

$$ayp^2 - x^2p - x(y + b) = 0,$$
$$ap^3 - 3xp - (y + b) = 0.$$

The values $x = 0$, $y = 0$, fulfil the former by its coefficients, and render the latter

$$ap^3 - b = 0, \because p = \sqrt[3]{\frac{b}{a}},$$

which giving but one real value of p, the point is not a multiple point.

Ex. 2. Let the equation of the curve be

$$y^4 - x^5 + x^4 + 3x^2y^2 = 0.$$

By differentiating, we find

$$\text{A} = 2y(2y^2 + 3x^2), \quad \text{B} = x(4x^2 - 5x^3 + 6y^2).$$

The only values of x and y which satisfy the equations $\text{A} = 0$, $\text{B} = 0$, as well as that of the curve, are $x = 0$, $y = 0$.

To determine p, let the successive differentiations be effected, and it will be found that the second and third differential equations will be satisfied by their coefficients, and that the fourth becomes

$$p^4 + 3p^2 + 1 = 0,$$

the roots of which being impossible, indicates a conjugate point.

Ex. 3. Let the equation of the curve be

$$x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$$

By differentiating, we find

$$\text{A} = 3ay(a + y), \quad \text{B} = 2x(a^2 - x^2).$$

The only values of xy which fulfil the conditions $\text{A} = 0$, $\text{B} = 0$, as well as the equation of the curve, are

$$\left.\begin{matrix} y = 0 \\ x = +a \end{matrix}\right\} \quad \left.\begin{matrix} y = 0 \\ x = -a \end{matrix}\right\} \quad \begin{matrix} y = -a, \\ x = 0. \end{matrix}$$

To determine the corresponding values of p, we proceed to the second differential equation, which gives

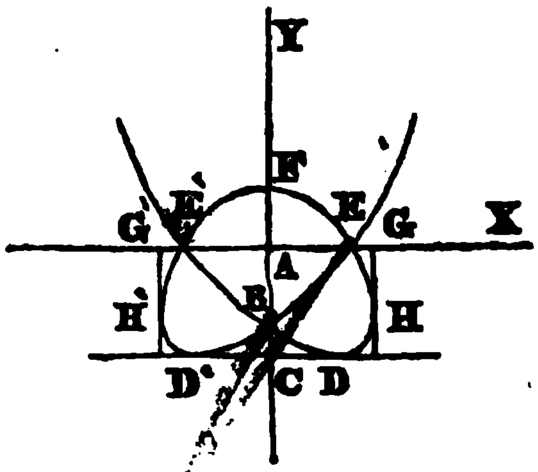
$$3a(a + 2y)p^2 + 2a^2 - 6x^2 = 0.$$

For the first and second points, therefore, $p = \pm \dfrac{2}{\sqrt{3}}$, and for the third $p = \pm \sqrt{\frac{2}{3}}$. The three corresponding points are therefore double points.

The condition $\text{B} = 0$, and the equation of the curve are also fulfilled by $x = 0$, $y = \frac{1}{3}a$, which values do not fulfil $\text{A} = 0$; therefore they determine a point at which the tangent

is parallel to the axis of x. The same conditions are also fulfilled by $x = \pm a$, $y = -\frac{3}{2}a$, which also determine tangents parallel to the axis of x.

The condition $A = 0$, and the equation of the curve are fulfilled by $y = -a$ and $x = \pm a\sqrt{2}$, which do not fulfil $B = 0$, $\because$ they indicate two points at which the tangents are parallel to the axis of y. To construct this curve, let AX and AY be the axes of co-ordinates.



Assume $AF = \frac{1}{2}a$, $AB = -a$, $AE = +a$, and $AE' = -a$, $AC = -\frac{3}{2}a$, $CD = +a$, $CD' = -a$, $AG = +a\sqrt{2}$, $AG' = -a\sqrt{2}$, $GH = -a$ and $G'H' = -a$. The curve is placed as represented in the diagram. The tangents at the double points B, E, E′, are determined by $p = \pm\sqrt{\frac{2}{3}}$ and $p = \pm\frac{2}{\sqrt{3}}$.

It is not necessary to multiply examples, as the student may easily supply himself with sufficient to illustrate the general theory. The following curves have triple points:

$$x^4 + 2ax^2y - ay^3 = 0,$$
$$y^4 - axy^2 + x^4 = 0,$$
$$y^4 + x^4 - 3ay^3 + 2bx^2y = 0.$$

SECTION XVI.

Of the singular points at which y *or any of its differential coefficients become infinite.*

(160.) We shall now proceed to investigate the figure of a curve at a point whose co-ordinates render the first or any subsequent differential coefficients infinite.

If the value assigned to x render y infinite, the first exponent of h in the development of y' must be negative (55.). In this case, as h is continually diminished, y' is continually increased; and when $h = 0$, y' becomes infinite. Thus it appears that a parallel to the axis of y corresponding to this value of x must be an asymptote.

If the origin of co-ordinates be removed to the point in question, then h becomes x, and the result immediately follows from Section XIV.

(161.) If the value of x render any of the differential coefficients infinite, rules have been already given for determining the successive exponents of h in the development of y' (55.). We shall not here, therefore, enter into any repetition of these methods, but assume the development of the form

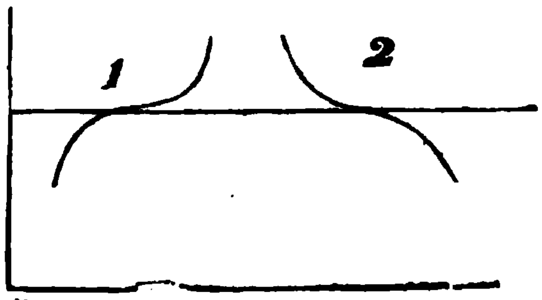
$$y' = y + \text{A}h^a + \text{B}h^b + \text{C}h^c \cdot \cdot \cdot \cdot$$

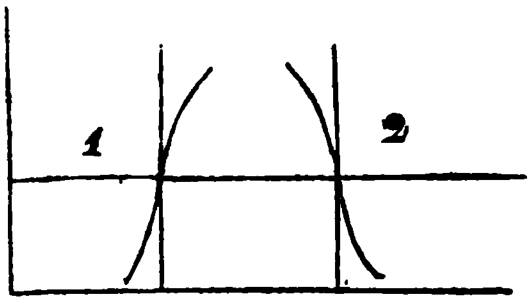
If none of the exponents a, b, c, $\cdot \cdot \cdot \cdot$ be a fraction with an even denominator, the value of y' is real, whether h be $+$ or $-$. Hence the curve extends on both sides of the ordinate y.

There are then two cases to examine, 1°. Where the numerator of the first exponent is odd, and 2°. Where it is even.

1°. If the numerator of a be odd, the sign of $\text{A}h^a$ changes with that of h, and consequently at different sides of the point yx, the curve lies at different sides of a parallel to the axis of x passing through the point.

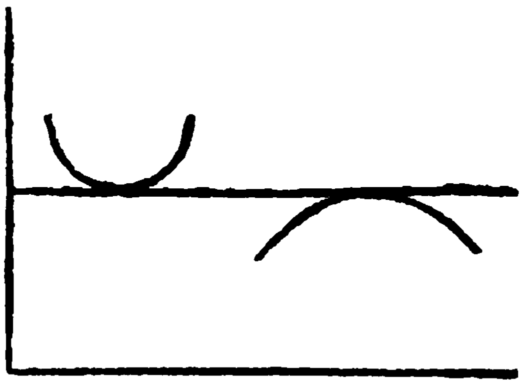
If in this case $a > 1 \; \frac{dy}{dx} = 0, \therefore (132.)$ the tangent is parallel to the axis of x. Hence there is an inflexion which is represented as in the annexed figures, the first when $\text{A} > 0$, and the second when $\text{A} < 0$.



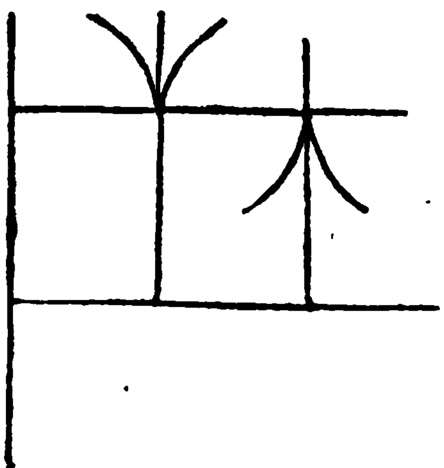


If $a < 1$, $\because \frac{dy}{dx}$ is infinite, and the tangent is parallel to the axis of y (132.). Since the curve extends on both sides of y, and crosses the parallel to the axis of x, the point must be an inflexion, as represented in the first figure when A [illegible], and in the second when A [illegible].

2°. Let the numerator of the first exponent be even. In this case the sign of $\text{A}h^a$ does not change with that of h, and since the denominator is supposed to be odd, there is but one real value; and since by diminishing h, the term $\text{A}h^a$ predominates over those which follow it (88.), $y' - y$ has the same sign for $x + h$ and $x - h$. Therefore, if a parallel to the axis of x be drawn through the point xy, the curve lies either above or below this parallel at both sides of the point according as A is [illegible] or [illegible]



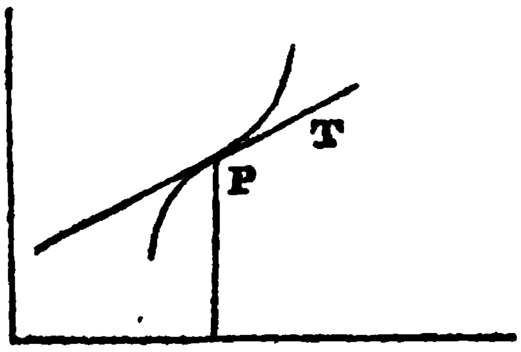
In this case, if $a > 1$, $\because \frac{dy}{dx} = 0$, $\because$ the parallel to the axis of x is a tangent, and the curve is as represented in the first or second figure, according as A is [illegible] or [illegible].



If $a < 1$, $\because \frac{dy}{dx}$ is infinite, $\because$ the tangent is parallel to the axis of y. Hence the figure of the curve at the point in question is as represented in the first figure if A [illegible], and in the second if A [illegible].

If the first exponent $a = 1$, and the second exponent have

an odd numerator, then the position of the tangent PT is determined by the value of A; and since the sign of the second term of the development changes with the sign of h, it follows that at different sides of the point P the curve lies at different sides of the tangent, as represented in this figure. Hence, in this case the point P is a point of inflexion.



The second differential coefficient in this case $= 0$, if the exponent $b > 2$, and is infinite if $b < 2$. Thus at a point of inflexion the second differential coefficient may be either nothing or infinite.

If $a = 1$, and the numerator of b be even, the succeeding exponents not having any even denominator, the point is marked by no peculiarity.

(162.) If amongst the exponents $a, b, c, \cdots\cdot$ is found a fraction with an even denominator, then a change in the sign of h changes y' from real to imaginary, or *vice versa* If $+ h$ render all the terms of the development which are affected by such exponents real, and $-h$ imaginary, the curve extends only on the positive side of y, and is excluded from the negative side; and if $- h$ render them real, and $+ h$ imaginary, it is excluded from the positive, and only extends upon the negative side.

If $+ h$ render some terms which are affected by such exponents imaginary, and $- h$ others, then the curve is excluded from both sides, and the point is a *conjugate point.*

If $+ h$ or $- h$ render all the terms whose exponents have even denominators, real, each of such terms will have two real values for every value of h, and therefore the number of branches of the curve emerging from the point in question will be double the number of combinations of such powers. The tangent to these branches will be de-

termined by the value of the lowest exponent of h. If it be > 1, the tangent is parallel to the axis of x, if < 1, it is parallel to the axis of y, and if $= 1$, its position is determined by the coefficient (132.).

(163.) Some particular cases will make this general principle more apparent.

1°. Let the lowest exponent of h be a fraction with an even denominator and ∵ with an odd numerator, and suppose this the only even denominator which occurs in the series. Then

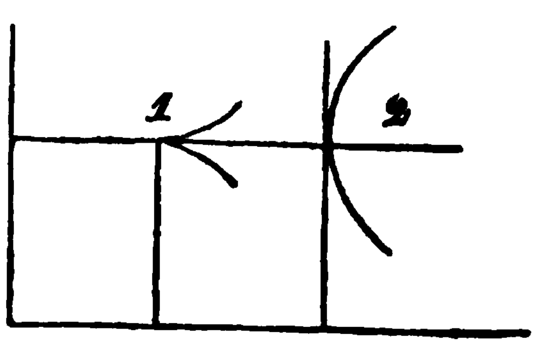
$$y' - y = \text{A}h^{\frac{m}{n}} + \text{B}h^b + \text{C}h^c \cdots$$

where n is by hyp. even.

In this case $+ h$ renders $h^{\frac{m}{n}}$ real, and $- h$ imaginary.

First. If A be real, for every positive value of h, there are two real values of $\text{A}h^{\frac{m}{n}}$ with different signs; and for every negative value of h, $\text{A}h^{\frac{m}{n}}$ is imaginary.

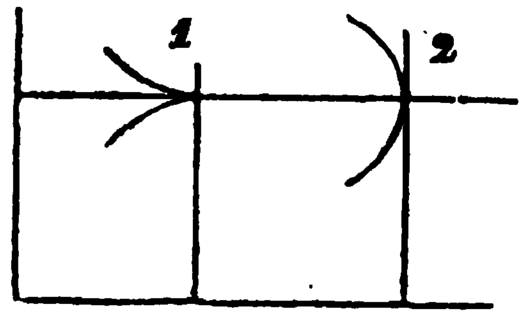
Also, if $\frac{m}{n} > 1$, $\frac{dy}{dx} = 0$, ∵ the tangent is parallel to the axis of x, and if $\frac{m}{n} < 1$, $\frac{dy}{dx}$ is infinite, and ∵ the tangent is parallel to the axis of y.



The first figure represents the curve at the point in question when $\frac{m}{n} > 1$, and the second when $\frac{m}{n} < 1$.

If A be such an imaginary quantity, that the term $\text{A}h^{\frac{m}{n}}$ is real for $- h$, it will be imaginary for $+ h$. Hence,

in this case the first figure is the case where $\frac{m}{n} > 1$, and the second where $\frac{m}{n} < 1$.



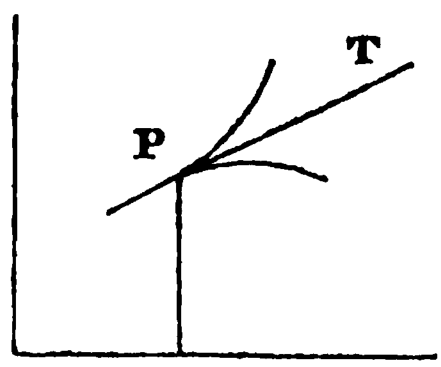
If A be any other species of imaginary quantity, $\text{A}h^{\frac{m}{n}}$ is imaginary for both $+h$ and $-h$, $\because$ the point is conjugate.

(164.) If the first exponent be an integer or a fraction whose denominator is odd, and the second exponent be a fraction $\frac{m}{n}$, whose denominator n is even, and that no other even denominator occurs in the series but n, then

$$y' - y = \text{A}h^a + \text{B}h^{\frac{m}{n}} + \text{C}h^c \cdot \cdot \cdot \cdot$$

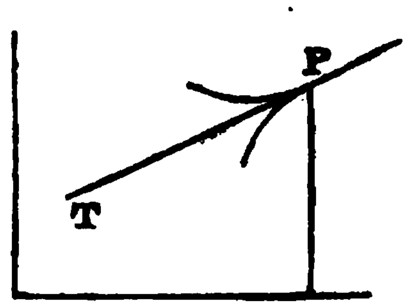
The position of the tangent is to be determined by the value of a as before.

If B be real, $+h$ renders $h^{\frac{m}{n}}$ real, and $-h$ imaginary.



Let PT be the tangent as determined by the term $\text{A}h^a$. Since there are two real values of $\text{B}h^{\frac{m}{n}}$ with different signs, the figure of the curve at the point P is this.

If B be such an imaginary quantity, that $\text{B}h^{\frac{m}{n}}$ is real for $-h$, and $\because$ imaginary for $+h$, the figure is this.



Such points where two branches lie at opposite sides of the common tangent are called *cusps of the first kind.*

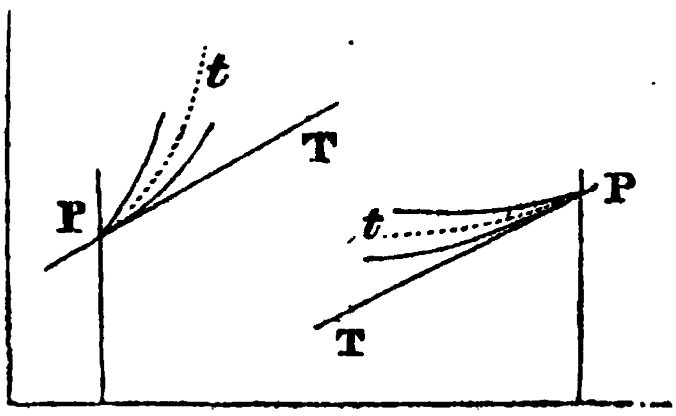
If B be an imaginary quantity of any other species, the point is conjugate.

(165.) If the only even denominator first occurs in the third term, the series is

$$y' - y = \text{A}h^a + \text{B}h^b + \text{C}h^{\frac{m}{n}} + \text{D}h^d \cdots$$

The position of the tangent being as before determined by $\text{A}h^a$, let the curve represented by

$$y' - y = \text{A}h^a + \text{B}h^b$$



be Pt. The branches of the curve evidently lie at different sides of this curve and at the same side of the tangent.

Hence if C be real, the curve is represented as in the first figure; and if C be an imaginary quantity which renders $\text{C}h^{\frac{m}{n}}$ real for $-h$, as in the second.

These are called *cusps of the second kind.*

It is evident that the branches in this case touch with contact of the second order.

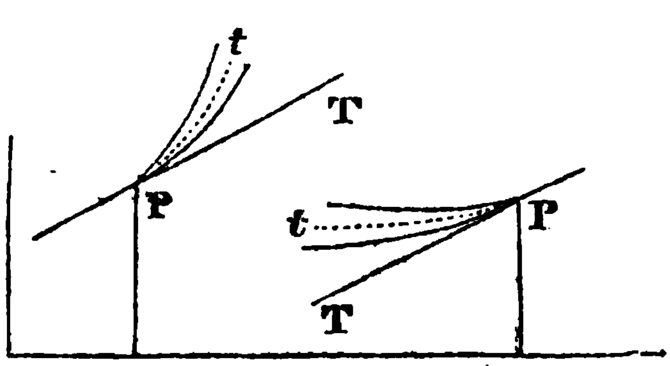
If C be an imaginary quantity of any other species the point is *conjugate.*

In general, if the first term of the series which has an even denominator be the rth,

$$y' - y = \text{A}h^a + \text{B}h^b \cdots \text{Q}h^q + \text{R}h^{\frac{m}{n}}.$$

Let Pt be the curve whose equation is

$$y' - y = \text{A}h^a + \text{B}h^b \cdots \text{Q}h^q.$$



It is evident, that if R be not an imaginary quantity which renders $\text{R}h^{\frac{m}{n}}$ imaginary, that the curve will have two branches emerging from P, which will lie at the same side of the tangent, and at different sides of the curve Pt.

Hence the point P will be a cusp of the second kind, and the figure will be as represented in either of the annexed diagrams.

In this case the two branches will have contact of the rth order.

If R be an imaginary quantity, which renders $\text{R}(\pm h)^{\frac{m}{n}}$ imaginary, the point will be conjugate.

It is unnecessary to pursue the examination of the particular cases further. The student will easily perceive the consequences of a combination of different even denominators in the exponents of the series in multiplying the branches of the curve passing through the given point, as well as the orders of their contact under different circumstances.

(166.) We shall conclude this investigation of singular points, which the importance and difficulty of the subject, as well as the obscurity of most elementary writers upon it, have induced us to render somewhat protracted, by giving the student some general directions for the discussion of a curve and the discovery of its figure and peculiarities. Let its equation be $\text{F}(xy) = 0$.

I. Solve, if possible, the equation $\text{F}(xy) = 0$ for either or both of the variables, and determine the limits of the real and imaginary values of each. This will frequently determine the extent of the curve or its limits in the directions of the axes of co-ordinates.

II. By differentiating the equation, having previously rendered it rational, if necessary, find the first differential equation

$$\text{A}p + \text{B} = 0,$$

in which the quantities A and B will be rational functions of the variables.

III. Find the values of xy which satisfy the equations

$$\text{F}(xy) = 0, \quad \text{B} = 0,$$

but which do not satisfy $\text{A} = 0$. These will determine

points at which the tangent is parallel to the axis of x, provided that the substitution of $x + h$, for x does not render y imaginary when h is assumed indefinitely small, x having the particular value determined by these equations. If this be the case, however, the point is conjugate.

IV. Find the values of xy which satisfy the equations

$$\text{F}(xy) = 0, \quad \text{A} = 0,$$

but which do not satisfy $\text{B} = 0$. These will determine the points at which the tangent is parallel to the axis of y, subject however to an exception similar to that in the former case, in which also the point is conjugate.

V. Find the values of xy, which satisfy the three equations

$$\text{F}(xy) = 0, \quad \text{A} = 0, \quad \text{B} = 0,$$

and let the value or values of p be determined as in (111.), and the species of the point will depend upon the number and nature of these values.

VI. Apply a similar investigation to the second and succeeding differential coefficients, and in these cases examine the exponents of h in the development of y', and singular points will be found by the principles established in this section.

VII. Examine the sign of the second differential coefficient, which will show the direction of curvature.

VIII. Find the points where the curve meets the axes of co-ordinates by determining the values of each variable when the other $= 0$.

IX. Let each variable be developed in a series of descending powers of the other. This will determine the species of the infinite branches, if the curve have any, and will show the asymptotes, curvilinear as well as rectilinear.

X. The evolute may be found, which frequently indicates remarkable properties in the curve itself.

(167.) Ex. 1. To determine the point of the curve whose equation is

$$(y - a)^5 = (x - b)^3,$$

where the tangent is parallel to the axis of y.

By differentiating

$$5(y - a)^4 dy = 3(x - b)^2 dx.$$

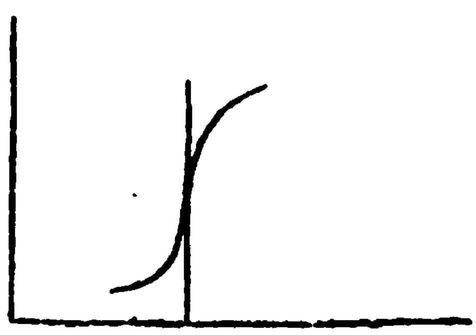
$$\therefore \frac{dy}{dx} = \frac{3}{5} \cdot \frac{(x-b)^2}{(y-a)^4}.$$

If $y = a$, this becomes infinite. The corresponding value of x is evidently $x = b$. Substituting $b + h$ for x, we find

$$(y' - a)^5 = h^3,$$

$$\therefore y' = a + h^{\frac{3}{5}}.$$

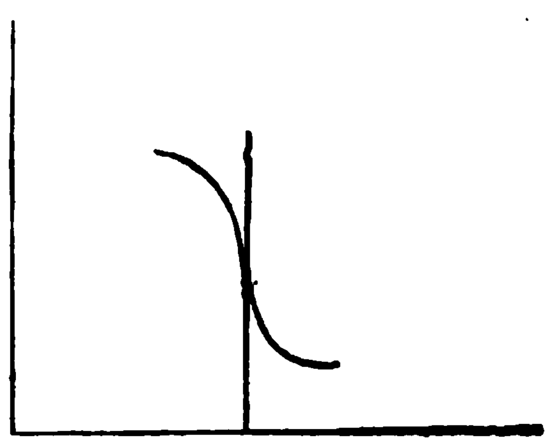
As both numerator and denominator are odd, and $\frac{3}{5} < 1$, the point whose co-ordinates are $y = a$ and $x = b$ is a point of inflexion represented thus.



Ex. 2. The curve represented by

$$(b - y)^5 = (x - a)^3,$$

may in like manner be shown to have a point of inflexion when $y = b$ and $x = a$, represented thus.



Ex. 3. To determine the point of the curve represented by the equation

$$(y - b)^3 = (x - a)^2,$$

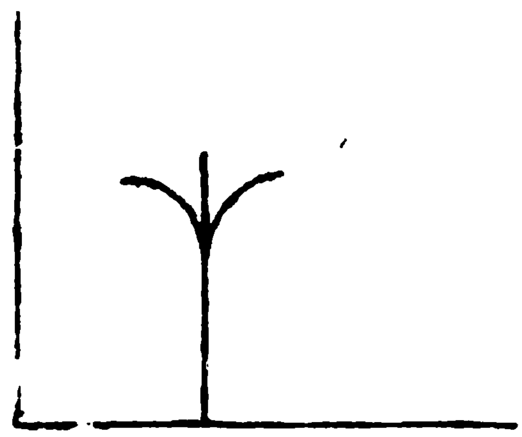
at which the tangent is parallel to the axis of y.

By differentiating,

$$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{x-a}{(y-b)^2} = \tfrac{2}{3}(y - b)^{-\frac{1}{2}},$$

$y = b$ renders this infinite, and the corresponding value of x is $x = a$. Let $x + h$ be substituted for x, and the result is

$$y = b + h^{\frac{2}{3}}.$$



Since the numerator of the exponent is even, and the denominator odd, and $\frac{2}{3} < 1$, the corresponding point is a cusp of the first kind represented thus.

Ex. 4. In like manner it may be shown that the curve represented by the equation

$$(b - y)^3 = (x - a)^2$$

has a cusp where $y = b$ and $x = a$, thus represented.

Ex. 5. To determine the point of the curve

$$(y - a - x)^4 = (x - b)^3,$$

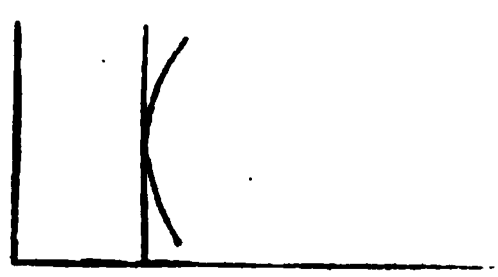
at which the tangent is parallel to the axis of y.

By differentiating

$$\frac{dy}{dx} = 1 + \tfrac{3}{4}(x - b)^{-\frac{1}{4}},$$

$x = b$ renders this infinite. Substituting $b + h$ for x in the original equation, we find

$$y = (a + b) + h^{\frac{3}{4}} + h.$$



Now since the numerator of the first exponent is odd, and the denominator even, and $\frac{3}{4} < 1$, the point is as in this figure.

Ex. 6. Let the equation be

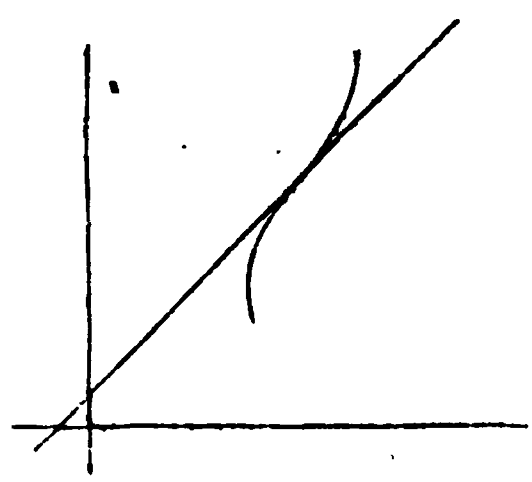
$$(y - x) = (x - a)^{\frac{5}{3}},$$

$$\therefore \frac{dy}{dx} = 1 + \tfrac{5}{3}(x - a)^{\frac{2}{3}},$$

$$\frac{d^2y}{dx^2} = \tfrac{5}{3} \cdot \tfrac{2}{3}(x - a)^{-\frac{1}{3}}.$$

If $x = a$, $\frac{dy}{dx} = 1$, and $\frac{d^2y}{dx^2}$ is infinite. In this case let $a + h$ be substituted for x in the equation, and we find

$$y = a + h + h^{\frac{5}{3}}.$$



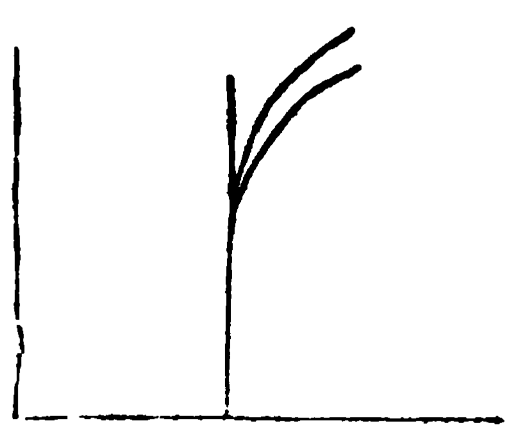
Since $\frac{dy}{dx} = 1$, the tangent is inclined to the axis of x at an angle of 45°; and since the numerator and denominator of the second exponent are both odd, the point is a point of inflexion.

Ex. 7. Let the equation be

$$y - a = (x - b)^{\frac{1}{3}} + (x - b)^{\frac{3}{4}}.$$

In this case $x = b$ renders all the differential coefficients infinite, and renders $y = a$. Let $b + h$ be substituted for x, $\therefore$

$$y = a + h^{\frac{1}{3}} + h^{\frac{3}{4}}.$$



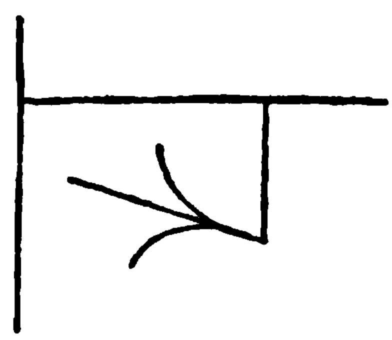
Since $\frac{1}{3} < 1$, the tangent is parallel to the axis of y; and since the denominator of the second exponent is even, the point is a cusp of the second kind.

Ex. 8. Let the equation be

$$(2y + x + 1)^2 = 2(1 - x)^5.$$

In this case $x = 1$ renders the third differential coefficient infinite. Substituting $1 + h$ for x, we find

$$y' = -1 - \tfrac{1}{2}h + (-h)^{\frac{5}{2}}.$$

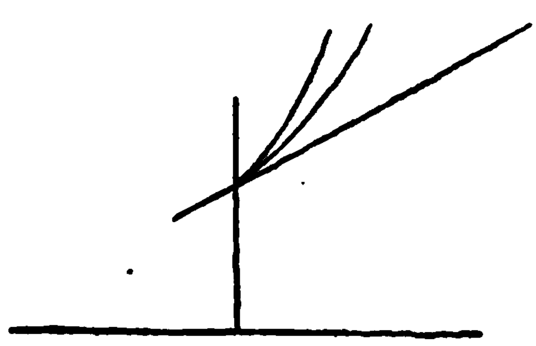


In this case it is necessary to take h negative, in order that y' may be real; and since the exponent of the first power of h is unity, the tangent is inclined to the axis of x at an angle whose tangent is $-\frac{1}{2}$. Also, since the exponent $\frac{5}{2}$ has an even denominator, the cusp is of the first kind.

Ex. 9. Let the equation be

$$y - a = x + bx^2 + cx^{\frac{5}{2}};$$

the third differential coefficient becomes infinite when $x = 0$ and $y = a$.



Substituting $0 + h$ for x, we find

$$y' = a + h + bh^2 + ch^{\frac{5}{2}};$$

by the principles established, this is a cusp of the second kind.

SECTION XVII.

On the application of the differential calculus to the geometry of curved surfaces.

(168.) A complete investigation of those properties of surfaces, which are discoverable by the aid of the differential calculus, would lead us into details inconsistent with the objects of the present treatise. We shall therefore in this section confine ourselves to a few of the most striking and

useful applications of the calculus to geometry of three dimensions. Students who are desirous of prosecuting the subject further will find it in its fullest details in the second volume of my Geometry.

(169.) An equation between three variables represents in general a surface. Any one of the variables (z), being considered as a function of the other two, and a point being assumed upon the plane xy, the corresponding point of the surface will be determined by the equation $F(xyz) = 0$.

If any value x' be given to x, the equation $F(x'yz) = 0$, represents the section of the surface by a plane parallel to the plane (yz), at the distance (x'). In like manner $F(xy'z) = 0$ and $F(xyz') = 0$ represent sections parallel to the planes xz and yx respectively.

If $F(xyz) = u = 0$, the partial differential equations of the first order,

$$\frac{du}{dx}dx + \frac{du}{dy}dy = 0,$$

$$\frac{du}{dy}dy + \frac{du}{dz}dz = 0,$$

$$\frac{du}{dz}dz + \frac{du}{dx}dx = 0,$$

are those of the sections parallel to the co-ordinate planes at the distances z, x, and y respectively. This is plain from the meaning of the notation (95), and from the preceding observations. From these equations the equations of tangents to those sections may be easily determined.

(170.) If z be considered as a function of x and y, and z' be what z becomes when x and y become $x + h$ and $y + k$, let z' be developed by Taylor's theorem in powers of h and k, the result will be

$$z' = z + A_{,}\frac{h}{1} + B_{,}\frac{k}{1} + A_2\frac{h^2}{1.2} + C_{,}hk + B_2\frac{k^2}{1.2} + A_3\frac{h^3}{1.2.3}$$
$$+ C_2\frac{h^2k}{1.2} + D_2\frac{hk^2}{1.2} + B_3\frac{k^3}{1.2.3} \cdot \cdot \cdot \cdot$$

Let the equation of another surface having a common point xyz with the proposed one be $\mathrm{F}'(xyz) = 0$, and let z'' be the value of z corresponding to $x + h$ and $y + k$, $\therefore$

$$z'' = z + a_{\prime}\frac{h}{1} + b_{\prime}\frac{k}{1} + a_2\frac{h^2}{1.2} + c_{\prime}\frac{hk}{1} + b_2\frac{k^2}{1.2} + a_3\frac{h^3}{1.2.3}$$
$$+ c_2\frac{h^2k}{1.2} + d_2\frac{hk^2}{1.2} + b_3\frac{k^3}{1.2.3} \dots$$

The coefficients of these series respectively being the successive differential coefficients (96).

By the reasoning used in (129), it follows that if

$$\mathrm{A}_{\prime} = a_{\prime}, \quad \mathrm{B}_{\prime} = b_{\prime},$$

no surface of which the first differential coefficients have values different from these can pass between them.

Hence, if the surface $\mathrm{F}(xyz) = 0$ be supposed given, and the constants of the equation $\mathrm{F}'(xyz) = 0$ be so assumed as to fulfil the above condition, no other surface $\mathrm{F}''(xyz) = 0$, of which the constants do not fulfil this condition, can pass between them. The surfaces are said in this case to touch with contact of the first order.

Again, if the constants of $\mathrm{F}'(xyz) = 0$ be so assumed that

$$\mathrm{A}_{\prime} = a_{\prime}, \quad \mathrm{B}_{\prime} = b_{\prime},$$
$$\mathrm{A}_2 = a_2, \quad \mathrm{C}_{\prime} = c_{\prime}, \quad \mathrm{B}_2 = b_2.$$

The two surfaces touch with contact of the second order, and so on.

(171.) It is obvious that in order that the surface $\mathrm{F}'(xyz)$ having a common point with the given surface, may at that point have contact of the first order, it is necessary that there should be at least two independent constants in its equation; in order to have contact of the second order, there must be five independent constants; and in order to have contact of the nth order, there must be $\frac{n(n+3)}{2}$ disposable constants.

(172.) The equation of a plane through a given point being of the form

$$(z - z') - p(x - x') - q(y - y') = 0.$$

It is plain that for it

$$a_{\prime} = p, \quad b_{\prime} = q.$$

Hence the equation of the tangent plane to a surface at the point $x'y'z'$ is

$$(z - z') - \frac{dz'}{dx'}(x - x') - \frac{dz'}{dy'}(y - y') = 0,$$

where $\frac{dz'}{dx'}$, $\frac{dz'}{dy'}$, are the values of the partial differential coefficient corresponding to the point of contact.

(173.) The equations of a right line perpendicular to this through the point $x'y'z'$ are

$$(y - y') + \frac{dz'}{dy'}(z - z') = 0,$$

$$(x - x') + \frac{dz'}{dx'}(z - z') = 0,$$

which are, therefore, the equations of the normal.

Let nx, ny, nz, be the angles under the normal and the axes of co-ordinates, and let $\text{K} = \sqrt{1 + \left(\frac{dz'}{dx'}\right)^2 + \left(\frac{dz'}{dy'}\right)^2}$ and $p = \frac{dz'}{dx'}$, $q = \frac{dz'}{dy'}$, $\because$

$$\cos. nx = \frac{p}{\text{K}}, \quad \cos. ny = \frac{q}{\text{K}}, \quad \cos. nz = \frac{1}{\text{K}}.$$

These are sometimes expressed otherwise.

If $u = \text{F}(xyz) = 0$,

$$\frac{du}{dx} \div \frac{du}{dz} = \frac{dz}{dx},$$

$$\frac{du}{dy} \div \frac{du}{dz} = \frac{dz}{dy}.$$

Hence by these substitutions, we find

$$\cos. nx = \frac{\frac{du'}{dx'}}{\text{K}'}, \quad \cos. ny = \frac{\frac{du'}{dy'}}{\text{K}'}, \quad \cos. nz = \frac{\frac{du'}{dz'}}{\text{K}'},$$

where $\text{K}'^2 = \left(\frac{du'}{dx'}\right)^2 + \left(\frac{du'}{dy'}\right)^2 + \left(\frac{du'}{dz'}\right)^2$.

(174.) Every line drawn in the tangent plane through the point of contact is a tangent to the curve; it is sometimes useful to know which of these lines is most inclined to the plane xy. This is evidently that which is drawn perpendicularly to the intersection of the tangent plane with the plane xy. The equation of this line may be found thus. Let $z = 0$ in the equation of the tangent plane, and the result

$$z' + p(x - x') + q(y - y') = 0$$

is the equation of the intersection of the tangent plane with the plane xy. The equation of a line through $x'y'$ perpendicular to this is

$$p(y - y') - q(x - x') = 0.$$

This is the projection of the sought line upon the plane xy, and, therefore, with the equation of the tangent plane represents that line.

PROP. LXII.

(175.) *To find the equation of a curve described upon a given surface, such, that the tangent to every point of it shall be the tangent of greatest inclination to the plane* xy.

By differentiating the equation of the projection of the tangent of greatest inclination upon the plane xy, we find

$$pdy - qdx = 0.$$

The quantities p and q are functions of xyz. The variable z being eliminated by means of the equation $\text{F}(xyz) = 0$ of the surface, the quantities p and q will become functions of

x and y alone. This, therefore, will be the differential equation of the sought curve. To find the primitive equation will require the aid of the integral calculus.

PROP. LXIII.

(176.) *To determine the sphere which touches a surface most intimately at any given point.*

Let the equation of the sphere be

$$(x - x'')^2 + (y - y'')^2 + (z - z'')^2 = \mathrm{R}^2,$$

xyz being the point of contact, $x''y''z''$ being the centre of the sphere, and therefore R its radius.

Since this equation involves but four disposable constants, the co-ordinates of the centre and the radius, it follows (171.), that the sphere does not allow of contact of the second degree.

The differential coefficients of z considered successively as a function of x and y, are

$$-\frac{x-x''}{z-z''}, \quad -\frac{y-y''}{z-z''},$$

in order that it may have contact of the first order, it is necessary that these should be equal to the differential coefficients p and q derived from the surface, ∴

$$(x - x'') + p(z - z'') = 0,$$
$$(y - y'') + q(z - z'') = 0.$$

These conditions are fulfilled by assuming the centre of the sphere upon the normal (173.), which is therefore the locus of the centres of all spheres which touch the surface at the proposed point.

The radius of the sphere is still undetermined, and therefore may be so assumed, that the sphere shall touch the surface in any proposed direction round the point with contact of the second degree. That is to say, if a section of the

surface be made by a plane passing through the normal in any given direction, a sphere may be found which will touch this section with contact of the second degree.

Let the projection of this section upon the plane xy be found, and let its differential equation be

$$dy = mdx,$$

m is therefore the tangent of the angle which the tangent to the projection of the curve on the plane xy makes with the axis of x.

As our inquiry is now limited to the points upon the plane xy, which are determined by the above equation, x and y cease to be independent variables, and their increments are connected by the relation

$$k = mh.$$

Substituting this value of k in the development of z', it becomes

$$z' = z + (\text{A}_1 + m\text{B}_1)\frac{h}{1} + (\text{A}_2 + 2m\text{C}_1 + m^2\text{B}_2)\frac{h^2}{1.2} + \cdots$$

Let the value of z, corresponding to $x + h$ in the equation of the sphere, be

$$\text{z} = z + (a_1 + mb_1)\frac{h}{1} + (a_2 + 2mc_1 + m^2b_2)\cdot\frac{h^2}{1.2} + \cdots$$

That these may have contact of the second order, it is necessary that

$$\text{A}_1 = a_1, \quad \text{B}_1 = b_1,$$

$$\text{A}_2 + 2\text{C}_1m + \text{B}_2m^2 = a_2 + 2c_1m + b_2m^2 \cdots\cdots [1].$$

Of these equations, the first two have been already shown to be those of the normal to the surface of the point. The quantities a_2, c_1, b_2, are the three differential coefficients of the second order derived from the equation of the sphere. Hence

$$a_2 = -\frac{1}{z - z''} + \frac{x - x''}{(z - z'')^2}\cdot\frac{dz}{dx},$$

$$c_1 = \frac{x-x''}{(z-z'')^2} \cdot \frac{dz}{dy} = \frac{(y-y'')}{(z-z'')^2} \cdot \frac{dz}{dx},$$

$$b_2 = -\frac{1}{z-z''} + \frac{y-y''}{(z-z'')^2} \cdot \frac{dz}{dy},$$

where $x''y''z''$ is the centre of the sphere. And since

$$-\frac{x-x''}{z-z''} = \frac{dz}{dx} = p,$$

$$-\frac{y-y''}{z-z''} = \frac{dz}{dy} = q.$$

We find

$$a_2 = -\frac{1+p^2}{z-z''},$$

$$c_1 = -\frac{pq}{z-z''},$$

$$b_2 = -\frac{1+q^2}{z-z''}.$$

Substituting these values in [1], the result is

$(A_2 + 2C_1m + B_2m^2)(z-z'') + (1+p^2) + 2pqm + (1+q^2)m^2 = 0.$

This equation determines the co-ordinate z'' of the centre of the sphere, which being known, the equations

$$x - x'' = -p(z - z''),$$

$$y - y'' = -q(z - z''),$$

determine $x''y''$.

The equation of the sphere being

$$R^2 = (x - x'')^2 + (y - y'')^2 + (z - z'')^2,$$

by substituting for $(x - x'')$, $(y - y'')$, their values, we find

$$R = (z - z'') \cdot \sqrt{1 + p^2 + q^2}.$$

The sphere thus determined has contact of the second order with any curve traced upon the given surface through the given point, provided that the projection of that curve upon the plane xy has its tangent through the projection of the given point inclined to the axis of x at an angle whose tangent is m.

PROP. LXIV.

(177.) *At a given point upon a curved surface, to determine upon the normal the limits between which the centres of all osculating spheres lie.*

This problem may be solved by finding the values of m, which render R a *maximum* and *minimum*.

To simplify the investigation, let the given point be assumed as origin, and the axes of x and y in the tangent plane, the normal being axis of z. In this position of the co-ordinate axes,

$$x = 0, \quad y = 0, \quad z = 0, \quad R = -z'', \quad p = 0, \quad q = 0.$$

Hence

$$R = -\frac{1+m^2}{A_2 + 2C_1 m + B_2 m^2},$$

which being differentiated, and its differential $= 0$, gives

$$m^2 + \frac{A_2 - B_2}{C_1} m - 1 = 0.$$

The roots of which determine the values of m, which give the greatest and least values of R.

Since the product of these roots $= 1$, the directions of greatest and least curvature are always at right angles. Geometry, vol. i. (34.).

The formulæ will be still further simplified by taking right lines in the directions of greatest and least curvature as axes of y and x. In this case, one value of m in the above equation becomes infinite, and the other $= 0$. Hence $C_1 = 0$, which reduces the formula for the radius of curvature corresponding to other values of m to

$$R = -\frac{1+m^2}{A_2 + B_2 m^2}.$$

Let R', R'', be the radii of the greatest and least osculating spheres. Their values are found by supposing m and $\frac{1}{m}$ successively $= 0$, $\because$

$$R' = -\frac{1}{A_2}, \quad R'' = -\frac{1}{B_2}.$$

Hence it appears that the radii of the greatest and least osculating spheres are the reciprocals of the partial differential coefficients of the second order.

PROP. LXV.

(178.) *To express the radius of any osculating sphere as a function of the radii of the greatest and least osculating spheres, and of the angles under the directions in which they osculate.*

By the last proposition,

$$R = -\frac{1+m^2}{A_2+B_2m^2}.$$

Let φ', φ'', be the angles under the directions in which the sphere whose radius is R, osculates, and the directions of the osculation of those whose radii are R', R''. Hence

$$m^2 = \frac{\cos.^2\varphi'}{\cos.^2\varphi''}, \quad A_2 = -\frac{1}{R'}, \quad B_2 = -\frac{1}{R''}.$$

Making these substitutions in the value of R, it becomes, after reduction,

$$R = \frac{R'R''}{R'\cos.^2\varphi'+R''\cos.^2\varphi''}.$$

Hence, if the radii of the greatest and least osculating spheres and the directions of their osculations be given, the radius of a sphere which osculates in any given direction may be found.

PROP. LXVI.

(179.) *To express the differential of the arc of a curve related to three rectangular axes.*

By reasoning exactly similar to that used in (126), we find

$$ds = \sqrt{dy^2 + dx^2 + dz^2}.$$

PROP. LXVII.

(180.) *To determine the equations of a tangent to a curve related to three rectangular co-ordinates.*

It is evident that the projections of the tangent upon the co-ordinate planes are the tangents to the projections of the curve upon these planes. Hence the equations of the tangent to a curve passing through the point $x'y'z'$, are

$$(z - z') - \frac{dz'}{dx'}(x - x') = 0,$$

$$(z - z') - \frac{dz'}{dy'}(y - y') = 0.$$

By substituting for the functions $\frac{dz'}{dx'}$, $\frac{dz'}{dy'}$, their values derived from the equations of the curve, the equations of a tangent through any given point may be found.

(181.) *Cor.* 1. Let tx, ty, tz, be the angles under the tangent and the axes of co-ordinates. It is evident that

$$\cos.tx = \frac{dx}{ds}, \quad \cos.ty = \frac{dy}{ds}, \quad \cos.tz = \frac{dz}{ds},$$

where $ds = \sqrt{dy^2 + dx^2 + dz^2}$.

(182.) *Cor.* 2. Hence the equation of the normal plane

through $x'y'z'$, or a plane perpendicular to the tangent, is

$$\frac{dy'}{ds}(y-y')+\frac{dx'}{ds}(x-x')+\frac{dz'}{ds}(z-z')=0,$$

$$\text{or } dy'(y-y')+dx'(x-x')+dz'(z-z')=0.$$

(183.) If the curve be not a plane curve, the successive tangents will not all lie in the same plane. The plane of three points of the curve, assumed indefinitely close to one another, is called the *osculating plane.*

Def. A curve, which is not all in the same plane, is called *a curve of double curvature.*

PROP. LXVIII.

(184.) *To determine the equation of the osculating plane at a given point upon a curve of double curvature.*

Let two points of the curve, indefinitely near to each other, be xyz and $x'y'z'$. The equation of a plane through these is

$$\text{A}(y-y')+\text{B}(x-x')+\text{C}(z-z')=0,$$

the point xyz being considered as variable, and $x'y'z'$ given.

In order that this may be the osculating plane, it should pass through two points contiguous to $x'y'z'$; it is necessary, also, that its first and second differentials should equal those of the curve. Let the equation be twice differentiated without assuming any independent variable, the results will be

$$\text{A}dy+\text{B}dx+\text{C}dz=0,$$

$$\text{A}d^2y+\text{B}d^2x+\text{C}d^2z=0.$$

Hence eliminating $\frac{\text{A}}{\text{C}}$ and $\frac{\text{B}}{\text{C}}$, we find

$$(dz'd^2x'-dx'd^2z')(y-y')+(dy'd^2z'-dz'd^2y')(x-x')+$$
$$(dx'd^2y'-dy'd^2x')(z-z')=0,$$

which is the equation of the sought plane.

(185.) *Cor.* Since the condition under which two planes intersect perpendicularly is, that the sum of the products of their corresponding coefficients $= 0$, the osculating and normal planes are at right angles; for (182.),

$$dy'(dz'd^2x' - dx'd^2z') + dx'(dy'd^2z' - dz'd^2y') + dz'(dx'd^2y' - dy'd^2x') = 0.$$

PROP. LXIX.

(186.) *To determine the radius of curvature to a given point in a curve related to three rectangular co-ordinates.*

This problem is most easily solved by considering the osculating circle as one passing through three consecutive points of the curve. Under this point of view, its plane must be the osculating plane; and as its radius passing through the given point must be normal to the curve, its centre must be in the intersection of the osculating and normal planes. If, therefore, $x'y'z'$ be the co-ordinates of its centre, they must satisfy the equations

$$dy(y - y') + dx(x - x') + dz(z - z') = 0,$$
$$\mathrm{Y}(y - y') + \mathrm{X}(x - x') + \mathrm{Z}(z - z') = 0,$$

where

$$\mathrm{Y} = dzd^2x - dxd^2z, \quad \mathrm{X} = dyd^2z - dzd^2y, \quad \mathrm{Z} = dxd^2y - dyd^2x.$$

All circles passing through the given point, and having their centres upon this right line, touch the curve. In order to determine that of most intimate contact, let the intersection of two consecutive normal planes be found, and the point where this intersection meets the right line thus determined will be the centre of the osculating circle. To effect this, let the equation of the normal plane be differentiated. Considering $x'y'x'$ as constant, which gives

$$d^2y(y - y') + d^2x(x - x') + d^2z(z - z') - ds^2 = 0,$$

where $ds^2 = dy^2 + dx^2 + dz^2$.

From this and the former equations, we find

$$x - x' = \frac{(\text{Y}dz - \text{Z}dy)ds^2}{\text{D}},$$

$$y - y' = \frac{(\text{Z}dx - \text{X}dz)ds^2}{\text{D}},$$

$$z - z' = \frac{(\text{X}dy - \text{Y}dx)ds^2}{\text{D}},$$

where

$$\text{D} = (\text{Y}dz - \text{Z}dy)d^2x + (\text{Z}dx - \text{X}dz)d^2y + (\text{X}dy - \text{Y}dx)d^2z.$$

Substituting these values in

$$\text{R}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

we obtain

$$\text{R}^2 = \frac{[(\text{X}dy - \text{Y}dx)^2 + (\text{Z}dx - \text{X}dz)^2 + (\text{Y}dz - \text{Z}dy)^2]ds^4}{\text{D}^2}.$$

But by the conditions

$$\text{X}dx + \text{Y}dy + \text{Z}dz = 0,$$
$$\text{X}^2 + \text{Y}^2 + \text{Z}^2 = \text{D}^2,$$

this gives

$$\text{R} = \frac{ds^3}{\sqrt{\text{X}^2 + \text{Y}^2 + \text{Z}^2}},$$

which is the value of the radius of curvature for a curve of double curvature.

If ds be taken as the independent variable, by differentiating the equation

$$ds^2 = dy^2 + dx^2 + dz^2,$$

we find

$$dyd^2y + dxd^2x + dzd^2z = 0.$$

This being squared and added to the value of D, gives

$$\text{D} = ds^2[(d^2y)^2 + (d^2x)^2 + (d^2z)^2].$$

Hence we find

$$R = \frac{ds^2}{\sqrt{(d^2y)^2 + (d^2x)^2 + (d^2z)^2}}.$$

See *Mecanique Celeste*, liv. i. chap. 2.

From the preceding formulæ, those of plane curves may easily be deduced.

PART II.

THE INTEGRAL CALCULUS.

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SECTION I.

Fundamental Principles.

(187.) THE object of the Integral Calculus is the determination of the primitive function or equation from which a given differential, or differential equation, may have been derived.

The primitive function is in this case called the *integral* of the proposed differential, and the process by which it is determined is called *integration*.

These terms "integral" and "integration" are taken from the infinitesimal calculus, and have their origin in notions of this science not consistent with the rigour and purity of mathematical reasoning. As, in the infancy of the science, differentials were considered as infinitely small quantities; so the original functions from which these differentials were obtained, were taken as the sums of the infinitely minute elements; and the process by which these primitive quantities were found from their differentials, was looked upon as the *summation* or *integration* of the small component parts, and the operation was expressed by the character $\int$

prefixed to the differential, thus, $\int x^m dx$, as the initial of the word "sum" or "summation." Modern mathematicians have reduced the science to more rigorous principles, but they have retained its former phraseology and symbols. Lagrange alone had the boldness to attempt a revolution, not only in the principles, but in the language and algorithm, or notation of the science; but he can scarcely be considered to have succeeded, at least in the latter, since all mathematicians, almost without an exception, adhere to the old symbols, though some of them use the principles and reasoning of Lagrange.

(188.) According to the language of Lagrange, the object of the integral calculus is to determine the *primitive* from the *derived* function; or, if applied to equations, to determine the *primitive equation* to a given *derived equation.*

According to the more commonly received phraseology, this branch of the science consists in the determination of the function, of which a given function is the differential coefficient, or the equation, which differentiated, would produce a given equation. As this process is exactly the reverse of that which forms the subject of the differential calculus, so the rules and methods to be used in it must be discovered by retracing our steps in that part of the science.

(189.) We shall, in the first instance, confine our attention to those differential coefficients which are functions of a single variable; and, as in the Differential Calculus, we shall successively consider the cases where they are algebraic and transcendental functions, algebraic functions being divided into, 1°. *rational and integral,* 2°. *rational and fractional,* and 3°. *irrational;* and transcendental into, 1°. *exponential,* 2°. *logarithmic,* and 3°. *circular.*

Before we enter upon the methods of integrating these

functions, it will be necessary to lay down a few principles immediately derivable from the differential calculus, and which may be considered among the fundamental principles of the integral calculus.

(190.) I. As an independent constant connected with any function disappears by differentiation, so it should reappear by integration. Thus, if $F'(x)$ be the differential coefficient of $F(x)$, it is also the differential coefficient of $F(x) + C$, C being a quantity independent of x. It is necessary, therefore, to add to every integral a constant, which is generally called the arbitrary constant, because its value cannot be derived from, and does not depend on, the differential coefficient, but must, if discoverable at all, be determined by other means.

(191.) II. If the value of the integral corresponding to any particular value of the variable happen to be known, the value of the arbitrary constant may be found. For, let the integral with the arbitrary constant be $F(x) + C$, and suppose that it is known that the value of the integral is A when the variable x is $= a$, $\therefore$ $A = F(a) + C$. Hence $C = A - F(a)$, $\therefore$ the integral is $F(x) - F(a) + A$.

If the value (a) of the variable which renders the integral $= 0$ be known, the integral is $F(x) - F(a)$.

(192.) III. As a constant factor of a function is not affected by differentiation (18.), so neither is it affected by integration. Thus, if $F'(x)$ be the differential coefficient of $F(x)$, $AF'(x)$ will be the differential coefficient of $AF(x)$, or, according to the symbols of the integral calculus,

$$\int A \cdot F'(x)dx = A\int F'(x)dx,$$

A being a quantity independent of x.

(193.) IV. As the differential of a function, which is the algebraical sum of several functions of the same variable, is the sum of the differentials of these functions (17.), so the integral of the sum of several differentials of functions of the same variable is the sum of the integrals of these differentials.

Thus,

$$\int \{ F'(x)dx + F''(x)dx - F'''(x)dx \} = \int F'(x)dx + \int F''(x)dx - \int F'''(x)dx.$$

(194.) V. As the differential of the product of two functions of the same variable is the sum of the alternate products of each function into the differential of the other, so the product of two functions is equal to the sum of the integrals of each function into the differential of the other. From this principle an important method of integration is deduced. Let XX' be two functions of x. Hence

$$XX' = \int X dX' + \int X' dX,$$

$$\therefore \int X dX' = XX' - \int X' dX.$$

By this equation the determination of one integral $\int X dX'$ is made to depend on another, viz. $\int X' dX$. Numerous instances of the efficacy of this method will appear hereafter. It is called *integration by parts*.

(195.) VI. A similar method may be deduced from the form for the differential of a fraction (23.).

$$d \cdot \frac{X}{X'} = \frac{dX}{X'} - \frac{X dX'}{X'^2},$$

$$\therefore \frac{X}{X'} = \int \frac{dX}{X'} - \int \frac{X dX'}{X'^2},$$

$$\therefore \int \frac{X dX'}{X'^2} = \int \frac{dX}{X'} - \frac{X}{X'}.$$

This, as in the former case, makes the integration of one differential depend on that of another; but it is not so generally useful a formula.

(196.) VII. As the differential coefficient of a power is found by diminishing the exponent by unity, and multiplying by the first exponent, so a differential, whose coefficient is a power, is integrated by increasing the exponent by unity, and dividing by the increased exponent. Thus, $\int A x^m dx = \frac{A x^{m+1}}{m+1} + C$, C being the arbitrary constant.

This rule extends to the integration of all differentials which can be reduced to the form $\text{A}x^m dx$.

Such is $\text{A}x^{n-1}(\text{B} + \text{C}x^n)^m dx$; for since $x^{n-1}dx = \frac{1}{n}d(x^n)$, if $x^n = z$, $\therefore x^{n-1}dx = \frac{1}{n}\cdot dz$, $\therefore \text{A}x^{n-1}(\text{B} + \text{C}x^n)^m dx =$

$$\frac{\text{A}}{n}(\text{B} + \text{C}z)^m dz.$$

Again, let $\text{B} + \text{C}z = y$, $\therefore \text{C}dz = dy$. Hence we find

$$\text{A}x^{n-1}(\text{B} + \text{C}x^n)^m dx = \frac{\text{A}}{n\text{C}}y^m dy,$$

$$\therefore \int \text{A}x^{n-1}(\text{B} + \text{C}x^n)^m dx = \frac{\text{A}}{n\text{C}}\cdot\frac{y^{m+1}}{m+1} + \text{D},$$

D being an arbitrary constant.

(197.) VIII. The preceding rule is subject to the exception $\int x^{-1}dx$, or $\int\frac{dx}{x}$; the value of this being $l'x + \text{C}$, C being, as usual, an arbitrary constant (190.). Under this case also come all those differentials which can be reduced by any transformations to the form $\frac{dx}{x}$. Such as $\frac{dx}{x+a} =$

$$\frac{d(x+a)}{x+a}, \therefore \int\frac{dx}{x+a} = l(x+a) + \text{C}.$$

Again,

$$\int\frac{5x^3dx}{3x^4+7} = \tfrac{5}{12}\int\frac{12x^3dx}{3x^4+7} = \tfrac{5}{12}\int\frac{d(3x^4+7)}{3x^4+7} = \tfrac{5}{12}l(3x^4+7) + \text{C}.$$

Here it may be remarked in general, that when an integral is a logarithm, the arbitrary constant may always be introduced as a factor of the quantity under the logarithm. For in

$$\int \text{F}'(x)dx = l[f(x)] + \text{C},$$

let the constant $\text{C} = l\text{A}$, $\therefore$

$$\int \text{F}'(x)dx = lf(x) + l\text{A} = l[\text{A}f(x)].$$

(198.) IX. From the differentials of an arc, considered

successively as a function of its sine, cosine, tangent, co-tangent, secant, cosecant, versed sine, and coversed sine, we deduce the following results.

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin.^{-1}x + \text{C},$$

$$-\int \frac{dx}{\sqrt{1-x^2}} = \cos.^{-1}x + \text{C},$$

$$\int \frac{dx}{1+x^2} = \tan.^{-1}x + \text{C},$$

$$-\int \frac{dx}{1+x^2} = \cot.^{-1}x + \text{C},$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec.^{-1}x + \text{C},$$

$$-\int \frac{dx}{x\sqrt{x^2-1}} = \text{cosec.}^{-1}x + \text{C},$$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \text{ver. sin.}^{-1}x + \text{C},$$

$$-\int \frac{dx}{\sqrt{2x-x^2}} = \text{cover. sin.}^{-1}x + \text{C}.$$

(199.) Some of the preceding integrals may be made more general by introducing a constant coefficient, and supplying a radius different from unity. The student will easily perceive that these modifications will give results of the following forms:

$$\int \frac{\text{A}dx}{\sqrt{\text{B}^2-\text{C}^2x^2}} = \frac{\text{A}}{\text{C}}\sin.^{-1}\frac{\text{C}x}{\text{B}} + \text{D},$$

$$-\int \frac{\text{A}dx}{\sqrt{\text{B}^2-\text{C}^2x^2}} = \frac{\text{A}}{\text{C}}\cos.^{-1}\frac{\text{C}x}{\text{B}} + \text{D},$$

$$\int \frac{\text{A}dx}{\text{B}^2+\text{C}^2x^2} = \frac{\text{A}}{\text{BC}}\tan.^{-1}\frac{\text{C}x}{\text{B}} + \text{D},$$

$$-\int \frac{\text{A}dx}{\text{B}^2+\text{C}^2x^2} = \frac{\text{A}}{\text{BC}}\cot.^{-1}\frac{\text{C}x}{\text{B}} + \text{D},$$

$$\int \frac{\mathrm{A}dx}{x\sqrt{\mathrm{B}^2x^2-\mathrm{C}^2}} = \frac{\mathrm{A}}{\mathrm{C}}\sec.^{-1}\frac{\mathrm{B}x}{\mathrm{C}} + \mathrm{D},$$

$$-\int \frac{\mathrm{A}dx}{x\sqrt{\mathrm{B}^2x^2-\mathrm{C}^2}} = \frac{\mathrm{A}}{\mathrm{C}}\mathrm{cosec.}^{-1}\frac{\mathrm{B}x}{\mathrm{C}} + \mathrm{D}.$$

In all of which D is the arbitrary constant.

SECTION II.

Of the integration of differentials, whose coefficients are rational functions of the variable.

(200.) All rational functions of x, and all which can be reduced to rational functions, are reducible to one or other of the following forms:

$$u = \mathrm{A}x^a + \mathrm{B}x^b + \mathrm{C}x^c \quad . \quad . \quad . \quad . \quad [1],$$

$$u = \frac{\mathrm{A}x^a + \mathrm{B}x^b + \mathrm{C}x^c \cdots\cdots}{\mathrm{A}'x^{a'} + \mathrm{B}'x^{b'} + \mathrm{C}'x^{c'} \cdots\cdots} \quad . \quad . \quad [2].$$

All the exponents in these series may be considered as integers; for if any fractional powers were found amongst them, they might be thus reduced to integral powers. Let the common denominator of all the fractional exponents be found, and let it be q; and let $y = x^{\frac{1}{q}}$, $\because$ $y^q = x$, and $y^m = x^{\frac{m}{q}}$; making these substitutions for x and its powers, the quantity becomes a rational function of y, and since $dx = qy^{q-1}dy$, it will continue rational when multiplied by the value of dx. This transformation, however, is not always necessary previously to integrating the formula.

(201.) We shall first consider the integration of udx when u has the form [1]. By (193.) and (196.),

$$\int udx = \frac{\mathrm{A}x^{a+1}}{a+1} + \frac{\mathrm{B}x^{b+1}}{b+1} + \frac{\mathrm{C}x^{c+1}}{c+1} \cdots\cdots + \mathrm{K},$$

K being an arbitrary constant. If, however, any of the exponents happen to be -1, the integral will be of the form $\text{M}lx$ (197.). This integration includes all cases of the form [1], and is applicable whatever be the nature of the exponents. They may be positive, negative, integral, or fractional, no previous transformation being necessary.

(202.) To this class may be referred all differentials, whose coefficients can be reduced to a finite series of the form [1], either by expansion, multiplication, or any other process. If the series [1] were supposed unlimited as to the number of its terms, all differentials, whose coefficients are capable of being developed in a series of powers of the variable, would be included. But, as this would not give the integral in a finite form, we shall not consider it here. It will become the subject of consideration hereafter. (Sect. VI.).

All differentials, whose coefficients have the forms,

$$\frac{\text{X}^m}{x^k}, \quad \frac{\text{X}^m\text{X}'^{m'}}{x^k}, \quad \frac{\text{X}^m\text{X}'^{m'}\text{X}''^{m''}}{x^k}, \text{ \&c. \&c.}$$

where m, m', m'', ... are positive integers, and X, X$'$, X$''$, ... functions of the form [1], the exponents a, b, c, ... being any numbers whatever, may be integrated by the above process. For they may be reduced to the form [1] by development and multiplication.

(203.) The integration of differentials, whose coefficients come under the form [2], presents greater difficulties. If any of the exponents be negative, they may be removed by multiplying both terms of the fraction by a power of x with the same positive exponent, and if any exponent be fractional, it may be made to disappear by the transformation explained in (200.).

Let the terms of the numerator and denominator be then arranged, so that the exponents shall descend. If the first exponent of the numerator be greater than, or equal to, that

of the denominator, the fraction may, by actual division, be resolved into two parts, one of the form [1], and the other of the form [2], the exponent a being less than a', and the exponents being arranged in descending order. The differential being thus resolved into two, the first is integrable by the method already explained. The second may be resolved into as many fractions, whose numerators are of the form $\text{A}x^a dx$ as there are terms in the numerator, and thus the problem is reduced to the integration of a differential, whose coefficient is of the form

$$\frac{\text{A}x^a}{\text{A}'x^{a'}+\text{B}'x^{b'}+\text{C}'x^{c'}\ldots},$$

the exponents being integral and positive, and $a' > a$.

(204.) Such a fraction may always (see note, page 183) be reduced to a series of fractions, each of which must come under some one of the following forms:

$$\int\frac{\text{M}dx}{x+a},\quad \int\frac{\text{M}dx}{(x+a)^n},\quad \int\frac{(\text{M}x+\text{N})dx}{x^2+a^2},\quad \int\frac{(\text{M}x+\text{N})dx}{(x^2+a^2)^n}.$$

Hence the problem will be solved in general when methods of integrating these four forms shall have been explained.

(205.) I. To integrate the first form, it is only necessary to observe, that $dx = d(x+a)$; and, since M is constant, by (192) and (197),

$$\int\frac{\text{M}dx}{x+a} = \text{M}l\text{C}(x+a),$$

C being an arbitrary constant.

(206.) II. In like manner the second formula is integrated by considering $dx = d(x+a)$ and $\frac{1}{(x+a)^n} = (x+a)^{-n}$.

Hence by (196.),

$$\int\frac{\text{M}dx}{(x+a)^n} = -\frac{\text{M}}{(n-1)(x+a)^{n-1}}.$$

(207.) III. In the third formula the integral may be resolved into two; thus,

$$\int \frac{\mathrm{M}x+\mathrm{N}}{x^2+a^2}dx = \int \frac{\mathrm{M}xdx}{x^2+a^2} + \int \frac{\mathrm{N}dx}{x^2+a^2}.$$

Since $2xdx = d(x^2) = d(x^2 + a^2)$, it is obvious that, neglecting the constant,

$$\int \frac{\mathrm{M}xdx}{x^2+a^2} = \tfrac{1}{2}\mathrm{M}l(x^2 + a^2).$$

And by (199.),

$$\int \frac{\mathrm{N}dx}{a^2+x^2} = \frac{\mathrm{N}}{a}\tan.^{-1}\frac{x}{a}.$$

Hence by combining these results, and supplying the constant,

$$\int \frac{\mathrm{M}x+\mathrm{N}}{x^2+a^2}dx = \tfrac{1}{2}\mathrm{M}l(x^2 + a^2) + \frac{\mathrm{N}}{a}\tan.^{-1}\frac{x}{a} + \mathrm{C}.$$

(208.) IV. The fourth formula may be resolved into two,

$$\int \frac{\mathrm{M}x+\mathrm{N}}{(x^2+a^2)^n}dx = \int \frac{\mathrm{M}xdx}{(x^2+a^2)^n} + \int \frac{\mathrm{N}dx}{(x^2+a^2)^n}.$$

The first is easily integrated by considering that $2xdx = d(x^2) = d(x^2 + a^2)$. Hence

$$\int \frac{\mathrm{M}xdx}{(x^2+a^2)^n} = \frac{\frac{1}{2}\mathrm{M}}{1-n}(x^2 + a^2)^{1-n}.$$

To integrate the second part, it will be necessary to have recourse to the method of indeterminate coefficients. Let

$$\int \frac{\mathrm{N}dx}{(x^2+a^2)^n} = \frac{\mathrm{K}x}{(x^2+a^2)^{n-1}} + \mathrm{L}\int \frac{dx}{(x^2+a^2)^{n-1}},$$

K and L being indeterminate quantities, whose values may be determined thus. Let this equation be differentiated and the result cleared of fractions, the factor dx being suppressed. Hence we find

$$\mathrm{N} = \mathrm{K}(x^2 + a^2) - 2\mathrm{K}(n - 1)x^2 + \mathrm{L}(x^2 + a^2).$$

Since these quantities must be equal, independently of x, we have

$$\mathrm{N} = (\mathrm{K} + \mathrm{L})a^2, \quad 3\mathrm{K} + \mathrm{L} - 2\mathrm{K}n = 0.$$

Hence determining K and L, and substituting their values, we find

$$\int \frac{\text{N}dx}{(x^2+a^2)^n} = \frac{\text{N}x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{(2n-3)\text{N}}{2(n-1)a^2} \cdot \int \frac{dx}{(x^2+a^2)^{n-1}}.$$

By repeating this process with the latter integral, we obtain an expression for it, depending on the integration of

$$\int \frac{dx}{(x^2+a^2)^{n-2}}.$$

And thus the process may be pursued until the exponent of $x^2 + a^2$ shall be reduced to unity, in which case the integral is reduced to Case III.

The preceding principles contain all that is necessary for the integration of differentials, whose coefficients are *rational.* It will be perceived that their integration, when *fractional,* depends on our power of resolving the denominator into simple or quadratic factors.

Note on Art. (204.).

(209.) The resolution of a rational fraction of the form

$$\frac{\text{U}}{\text{V}} = \frac{\text{A} + \text{B}x + \text{C}x^2 \cdot \cdot \cdot \cdot \text{M}x^{m-1}}{\text{A}' + \text{B}'x + \text{C}'x^2 \cdot \cdot \cdot \cdot \text{M}'x^m}$$

into a series of fractions of the forms given in (204.), being necessary for the integration of rational fractional functions, we shall here explain a method of effecting this resolution.

First, It is necessary to show that the denominator is always capable of being resolved into real factors of the forms,

$$\text{I. } (x + a), \qquad \text{II. } (x + a)^n,$$
$$\text{III. } (x^2 + a^2), \qquad \text{IV. } (x^2 + a^2)^n.$$

I. If the roots of the equation

$$\text{V} = \text{A}' + \text{B}'x + \text{C}'x^2 \cdot \cdot \cdot \cdot \text{M}'x^m = 0$$

be all real and unequal, it may be resolved into simple and real factors of the form $(x + a)$.

II. If there be any number n of real and equal roots, there will be a factor of the form $(x + a)^n$.

III. If there be a pair of imaginary roots, there will be a factor of the form $(z^2 + pz + q)$, $p^2 - 4q$ being a negative quantity. By substituting $x - \frac{p}{2}$ for z, the form becomes $x^2 - \frac{p^2}{4} + q$; now since $p^2 - 4q < 0$, $\because$ $-\frac{p^2}{4} + q > 0$, let it be expressed by a^2; the form becomes $x^2 + a^2$, which is the required form.

IV. If there be n pairs of equal imaginary roots, there will be a factor of the form $(z^2 + pz + q)^n$, $p^2 - 4q$ being negative. This, as before, may be reduced to the form $(x^2 + a^2)^n$.

(210.) Let us first suppose, that by the resolution of the equation in (I.) its several roots are obtained. If they be real and unequal, let any one of them be $-a$, then $x + a$ is a real simple factor of the denominator. Let

$$\frac{\text{V}}{x+a} = \text{Q},$$

it is evident that Q is an integral and rational quantity, the highest exponent of x in it being less than the highest exponent in V. Hence let

$$\frac{\text{U}}{\text{V}} = \frac{\text{A}}{x+a} + \frac{\text{P}}{\text{Q}},$$

A and P being undetermined; but A being independent of x, and P a rational function of x.

Since $\text{V} = (x + a)\text{Q}$, $\because$

$$\text{U} = \text{AQ} + \text{P}(x + a).$$

In this equation let $x = -a$, and let the corresponding values of the functions U and Q be u and q. Hence

$$\text{A} = \frac{u}{q},$$

and

$$P = \frac{U - AQ}{x + a},$$

which, since A has been determined, is known by actual division.

This method cannot fail, if, as has been supposed, the equation $V = 0$ admits no other root $= -a$, for in that case, $x = -a$ cannot render $Q = 0$, and, therefore, renders A finite and determinate, except when $-a$ happens to be a root of the equation $U = 0$, in which case $A = 0$.

Since the exponent of the highest power of x in Q and U is at least one less than in V, it is evident that the exponent of the highest power in P is at least two less than in V. Hence, another factor $x + a'$ being assumed, we can find

$$\frac{P}{Q} = \frac{A'}{x + a'} + \frac{R}{S},$$

provided that $x + a'$ is not one of several equal factors. By proceeding thus, the partial fractions corresponding to all the simple, real, and unequal factors of V may be determined, so that we shall have

$$\frac{U}{V} = \frac{A}{x + a} + \frac{A'}{x + a'} + \cdots\cdots + \frac{P'}{Q'},$$

Q' being a rational function of x, in which the highest exponent cannot exceed $m - n$, n being the number of simple, real, and unequal factors, and P' being likewise a rational function of x, in which the highest exponent of x cannot exceed $m - n - 1$. As all the real and unequal factors of V have been disposed of, Q' can only admit factors of the forms II., III., and IV.

(211.) We shall therefore now explain a method of finding the partial fractions which correspond to real factors of V of the form $(x + a)^n$. Let

$$\frac{U}{V} = \frac{A}{(x+a)^n} + \frac{A_1}{(x+a)^{n-1}} + \frac{A_2}{(x+a)^{n-2}} \cdots + \frac{A_{n-1}}{x+a} + \frac{P}{Q}.$$

By reducing these to the same denominator, we find

$$\text{U}=\text{Q}[\text{A}+\text{A}_{\prime}(x+a)\cdots\text{A}_{n-1}(x+a)^{n-1}]+\text{P}(x+a)^n,$$

$$\text{P}=\frac{\text{U}-\text{Q}[\text{A}+\text{A}_{\prime}(x+a)\cdots\text{A}_{n-1}(x+a)^{n-1}]}{(x+a)^n}.$$

Since P must be an integral function of x, the numerator of this expression must be divisible by $(x+a)^n$, and $\therefore$ it becomes $=0$ when $x=-a$. But it is obviously reduced in this case to $\text{U}-\text{QA}$. Let u and q be what U and Q become when $x=-a$; hence $u-\text{A}q=0$,

$$\therefore \text{A}=\frac{u}{q}.$$

Hence the quantity $\text{U}-\text{QA}$ becomes $\text{U}-\frac{u}{q}\text{Q}$. Now since this is divisible by $x+a$, let the quote be U$'$, so that

$$\text{U}-\text{Q}\frac{u}{q}=\text{U}'(x+a),$$

$$\text{P}=\frac{\text{U}'-\text{Q}[\text{A}_{\prime}+\text{A}_2(x+a)+\cdots\text{A}_{n-1}(x+a)^{n-2}]}{(x+a)^{n-1}}.$$

By applying a similar process to this fraction, $\text{A}_{\prime}$ may be determined, and similarly all the other numerators, so that the partial fractions corresponding to the case of equal factors become all known.

(212.) Methods nearly the same may be applied to the case where the equation $\text{V}=0$ has imaginary roots. By the transformation indicated in (209.) III. and IV., the denominator will be divisible by a factor of the form x^2+a^2, or $(x^2+a^2)^n$.

If it be divisible by a factor of the first kind, let

$$\frac{\text{U}}{\text{V}}=\frac{\text{A}x+\text{B}}{x^2+a^2}+\frac{\text{P}}{\text{Q}},$$

$$\therefore \text{U}=\text{Q}(\text{A}x+\text{B})+\text{P}(x^2+a^2)$$

Since P must be a rational function of x, $\text{U}-\text{Q}(\text{A}x+\text{B})$ must be divisible by x^2+a^2, and therefore ought to become $=0$ when $x=a\sqrt{-1}$.

When $a\sqrt{-1}$ is substituted for x in U and Q, they must

assume the forms $u + u'\sqrt{-1}$, $q + q'\sqrt{-1}$. And therefore we have

$$u + u'\sqrt{-1} - (q + q'\sqrt{-1})(\mathrm{A}a\sqrt{-1} + \mathrm{B}) = 0,$$
$$\therefore u - \mathrm{B}q + \mathrm{A}aq' + \sqrt{-1}(u' - \mathrm{A}aq - \mathrm{B}q') = 0.$$

And since the real and imaginary parts must severally $= 0$,

$$u - \mathrm{B}q + \mathrm{A}aq' = 0,$$
$$u' - \mathrm{B}q' - \mathrm{A}aq = 0,$$

which equations are sufficient to determine A and B.

(213.) Finally, we shall examine the case where V has several equal pairs of imaginary roots, and therefore, after transformation, admits a factor of the form $(x^2 + a^2)^n$.

Let

$$\frac{\mathrm{U}}{\mathrm{V}} = \frac{\mathrm{A}x + \mathrm{B}}{(x^2 + a^2)^n} + \frac{\mathrm{A}_1 x + \mathrm{B}_1}{(x^2 + a^2)^{n-1}} \cdots + \frac{\mathrm{P}}{\mathrm{Q}}.$$

$$\mathrm{P} = \frac{\mathrm{U} - \mathrm{Q}[(\mathrm{A}x + \mathrm{B}) + (\mathrm{A}_1 x + \mathrm{B}_1)(x^2 + a^2) \cdots (\mathrm{A}_{n-1}x + \mathrm{B}_{n-1}(x^2 + a^2)^{n-1}]}{(x^2 + a^2)^n}$$

Since P is a rational and integral function of x, $(x^2 + a^2)^n$ must divide the numerator, and therefore it becomes $= 0$ when $x = a\sqrt{-1}$.

By this substitution, let U become $u \pm \sqrt{-1} \cdot u'$, and Q, $q \pm \sqrt{-1} \cdot q'$, $\therefore$

$$u + \sqrt{-1} \cdot u' - \{q + \sqrt{-1} \cdot q'\} \cdot \{\mathrm{A}a\sqrt{-1} + \mathrm{B}\} = 0,$$

which are sufficient to determine A and B as before.

Having thus found the values of A and B, upon substituting them in the numerator of P, the term $\mathrm{U} - \mathrm{Q}(\mathrm{A}x + \mathrm{B})$ becomes divisible by $x^2 + a^2$. Let the quotient be U', $\therefore$

$$\mathrm{P} = \frac{\mathrm{U}' - \mathrm{Q}[\mathrm{A}_1 x + \mathrm{B}_1 + (\mathrm{A}_2 x + \mathrm{B}_2)(x^2 + a^2) \cdots\cdots]}{(x^2 + a^2)^{n-1}}.$$

The values of A_1, B_1, may hence be deduced by a process similar to that by which A and B were obtained.

SECTION III.

Praxis on the integration of differentials, whose coefficients are rational functions of the variable.

(214.) Ex. 1. Let $udx = \frac{\text{A}dx}{x^2-a^2}$, $\therefore \int udx = \text{A}\int\frac{dx}{x^2-a^2}$.

Let

$$\frac{1}{x^2-a^2} = \frac{\text{M}}{x-a} + \frac{\text{N}}{x+a},$$

$$\therefore 1 = (\text{M}+\text{N})x + (\text{M}-\text{N})a,$$

$$\therefore \text{M}+\text{N} = 0,\ \ \text{M}-\text{N} = \frac{1}{a},\ \therefore \text{M} = \frac{1}{2a},\ \ \text{N} = -\frac{1}{2a}$$

$$\therefore \frac{1}{x^2-a^2} = \frac{1}{2a}\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\},$$

$$\therefore \int\frac{\text{A}dx}{x^2-a^2} = \frac{\text{A}}{2a}\left\{l(x-a) - l(x+a)\right\} = \frac{\text{A}}{2a}l.\frac{x-a}{x+a}\text{*}.$$

Ex. 2. Let $udx = \frac{\text{A}dx}{x^2-5x+6}$. Since $x^2 - 5x + 6 = (x-2)(x-3)$, $\therefore$

$$\frac{1}{x^2-5x+6} = \frac{\text{M}}{x-2} + \frac{\text{N}}{x-3},$$

$$\therefore 1 = (\text{M}+\text{N})x - 3\text{M} - 2\text{N},$$

$$\therefore \text{M}+\text{N} = 0,\ \ 3\text{M}+2\text{N} = -1,$$

$$\therefore \text{M} = -1,\ \ \text{N} = 1,$$

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* It is to be understood that the arbitrary constant is omitted in the examples. It must of course be supplied in particular cases where it can be determined.

$$\therefore \int \frac{\text{A}dx}{x^2-5x+6} = \int \frac{-\text{A}dx}{x-2} + \int \frac{\text{A}dx}{x-3} = \text{A}\left\{ l(x-3) - l(x-2) \right\},$$

$$\therefore \int \frac{\text{A}dx}{x^2-5x+6} = \text{A}l\frac{x-3}{x-2}.$$

Ex. 3. Let $udx = \dfrac{(2-4x)dx}{x^2-x-2}$. Hence

$$\int udx = \int \frac{2dx}{2-x} - \int \frac{2dx}{x+1},$$

$$\therefore \int \frac{(2-4x)dx}{x^2-x-2} = -2\left\{ l(x-2) + l(x+1) \right\} =$$

$$-2l\cdot(x^2-x-2).$$

Ex. 4. Let $udx = \int \dfrac{xdx}{x^3-1}$. The factors of the denominator are $x-1$ and x^2+x+1. By resolving the fraction into two by the method of indeterminate coefficients, we find

$$\int \frac{xdx}{x^3-1} = \tfrac{1}{3}\int \frac{dx}{x-1} - \tfrac{1}{3}\int \frac{(x-1)dx}{x^2+x+1},$$

$$\therefore \int \frac{xdx}{x^3-1} = \tfrac{1}{3}l\,(x-1) - \tfrac{1}{3}\int \frac{(x-1)dx}{(x+\frac{1}{2})^2+\frac{3}{4}}.$$

Let $x+\frac{1}{2} = z$, $\frac{3}{4} = a^2$,

$$\therefore dx = dz, \quad x-1 = z-\tfrac{3}{2},$$

$$\therefore \int \frac{(x-1)dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \int \frac{zdz}{z^2+a^2} - \tfrac{3}{2}\int \frac{dz}{z^2+a^2}$$

$$= \tfrac{1}{2}l(z^2+a^2) - \frac{3}{2a}\tan.^{-1}\frac{z}{a};$$

restoring the values of z and a, we find

$$\int \frac{(x-1)dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = l\sqrt{x^2+x+1} - \sqrt{3}\tan.^{-1}\frac{2x+1}{\sqrt{3}},$$

$$\therefore \int \frac{xdx}{x^3-1} = \tfrac{1}{3}\left\{ l\frac{x-1}{\sqrt{x^2+x+1}} + \sqrt{3}\tan.^{-1}\frac{2x+1}{\sqrt{3}} \right\}.$$

Ex. 5. Let $udx = \dfrac{(x^2-x+1)dx}{x^3+x^2+x+1}$. The denominator in this case may be resolved into the factors $x+1$ and x^2+1, and thence

$$\int udx = \tfrac{3}{2}\int\frac{dx}{x+1} - \tfrac{1}{2}\int\frac{xdx}{x^2+1} - \tfrac{1}{2}\int\frac{dx}{x^2+1},$$

$$\therefore \int udx = \tfrac{3}{2}l(x+1) - \tfrac{1}{4}l(x^2+1) - \tfrac{1}{2}\tan.^{-1}x,$$

$$\int udx = l\frac{(x+1)^{\frac{3}{2}}}{(x^2+1)^{\frac{1}{4}}} - \tfrac{1}{2}\tan.^{-1}x.$$

Ex. 6. Let $udx = \dfrac{x^4+2x^3+3x^2+3}{(x^2+1)^3}dx$, by (213.),

$$\therefore udx = \frac{-2xdx}{(x^2+1)^3} + \frac{dx}{(x^2+1)^3} + \frac{2xdx}{(x^2+1)^2} + \frac{dx}{(x^2+1)^2} + \frac{dx}{x^2+1}.$$

And since

$$\int\frac{-2xdx}{(x^2+1)^3} = \frac{1}{2(x^2+1)^2},\quad \int\frac{2xdx}{(x^2+1)^2} = -\frac{1}{x^2+1}.$$

Also,

$$\int\frac{dx}{(x^2+1)^3} = \frac{x}{4(x^2+1)^2} + \tfrac{3}{4}\int\frac{dx}{(x^2+1)^2},$$

$$\int\frac{dx}{(x^2+1)^2} = \frac{x}{2(x^2+1)} + \tfrac{1}{2}\int\frac{dx}{x^2+1},$$

$$\int\frac{dx}{x^2+1} = \tan.^{-1}x.$$

Hence by combining these results, we find

$$\int udx = \frac{x+2}{4(x^2+1)^2} + \frac{7x-8}{8(x^2+1)} + \tfrac{15}{8}\tan.^{-1}x.$$

Ex. 7. Let $udx = \dfrac{xdx}{a+bx}$. By division, we find

$$udx = \frac{dx}{b} - \frac{a}{b}\cdot\frac{dx}{a+bx},$$

$$\therefore \int udx = \frac{x}{b} - \frac{a}{b^2}l(a+bx).$$

Ex. 8. Let $udx = \frac{xdx}{(a+bx)^3}$. Let $z = a + bx$,

$$\therefore dz = bdx.$$

Multiplying both numerator and denominator by b^2,

$$\int udx = \frac{1}{b^2}\int \frac{(z-a)dz}{z^3} = \frac{1}{b^2}\left\{-\frac{1}{z}+\frac{a}{2z^2}\right\},$$

$$\therefore \int udx = -\frac{1}{2b^2}\cdot\frac{a+2bx}{(a+bx)^2}.$$

Ex. 9. Let $udx = \frac{x^m dx}{(a+bx)^2}$; if $a + bx = \text{x}$,

The following integrations may be easily effected:

$$\int \frac{dx}{\text{x}^2} = -\frac{1}{b\text{x}},$$

$$\int \frac{xdx}{\text{x}^2} = \frac{a}{b^2\text{x}} + \frac{1}{b^2}\log.\ \text{x},$$

$$\int \frac{x^2dx}{\text{x}^2} = \left(\frac{x^2}{b} - \frac{2a^2}{b^2}\right)\frac{1}{\text{x}} - \frac{2a}{b^3}\log.\ \text{x},$$

$$\int \frac{x^3dx}{\text{x}^2} = \left(\frac{x^3}{2b} - \frac{3ax^2}{2b^2} + \frac{3a^3}{b^4}\right)\frac{1}{\text{x}} + \frac{3a^2}{b^4}\log.\ \text{x},$$

$$\int \frac{x^4dx}{\text{x}^2} = \left(\frac{x^4}{3b} - \frac{2ax^3}{3b^2} + \frac{2a^2x^2}{b^3} - \frac{4a^4}{b^5}\right)\frac{1}{\text{x}} - \frac{4a^3}{b^5}\log.\ \text{x}.$$

Ex. 10. Let $udx = \frac{x^m dx}{(a+bx)^3}$,

$$a + bx = \text{x},$$

$$\int \frac{dx}{\text{x}^3} = -\frac{1}{2b\text{x}^2},$$

$$\int \frac{xdx}{\text{x}^3} = -\left(\frac{x}{b} + \frac{a}{2b^2}\right)\frac{1}{\text{x}^2},$$

$$\int \frac{x^2dx}{\text{x}^3} = \left(\frac{2ax}{b^2} + \frac{3a^2}{2b^3}\right)\frac{1}{\text{x}^2} + \frac{1}{b^3}\log.\ \text{x},$$

$$\int \frac{x^3dx}{\text{x}^3} = \left(\frac{x^3}{b} - \frac{6a^2x}{b^3} - \frac{9a^3}{2b^4}\right)\frac{1}{\text{x}^2} - \frac{3a}{b^4}\log.\ \text{x},$$

$$\int \frac{x^4dx}{\text{x}^3} = \left(\frac{x^4}{2b} - \frac{2ax^3}{b^2} + \frac{12a^3x}{b^4} + \frac{9a^4}{b^5}\right)\frac{1}{\text{x}^2} + \frac{6a^2}{b^5}\log.\ \text{x}.$$

Ex. 11. Let $udx = \dfrac{dx}{x^m(a+bx)^3}$,

$$a + bx = \mathrm{X},$$

$$\int\frac{dx}{x\mathrm{X}^3}=\left(\frac{3}{2a}+\frac{bx}{a^2}\right)\frac{1}{\mathrm{X}^2}-\frac{1}{a^3}\log.\frac{\mathrm{X}}{x},$$

$$\int\frac{dx}{x^2\mathrm{X}^3}=\left(-\frac{1}{ax}-\frac{9b}{2a^2}-\frac{3b^2x}{a^3}\right)\frac{1}{\mathrm{X}^2}+\frac{3b}{a^4}\log.\frac{\mathrm{X}}{x},$$

$$\int\frac{dx}{x^3\mathrm{X}^3}=\left(-\frac{1}{2ax^2}+\frac{2b}{a^2x}+\frac{9b^2}{a^3}+\frac{6b^3x}{u^4}\right)\frac{1}{\mathrm{X}^2}-\frac{6b^2}{a^5}\log.\frac{\mathrm{X}}{x}.$$

Ex. 12. Let $udx = \dfrac{dx}{x^m(u+bx)^4}$,

$$a + bx = \mathrm{X},$$

$$\int\frac{dx}{x\mathrm{X}^4}=\left(\frac{11}{6a}+\frac{5bx}{2a^2}+\frac{b^2x^2}{a^3}\right)\frac{1}{\mathrm{X}^3}-\frac{1}{a^4}\log.\frac{\mathrm{X}}{x},$$

$$\int\frac{dx}{x^2\mathrm{X}^4}=\left(-\frac{1}{ax}-\frac{22b}{3a^2}-\frac{10b^2x}{a^3}-\frac{4b^3x^2}{a^4}\right)\frac{1}{\mathrm{X}^3}+\frac{4b}{a^5}\log.\frac{\mathrm{X}}{x},$$

$$\int\frac{dx}{x^3\mathrm{X}^4}=\left(-\frac{1}{2ax^2}+\frac{5b}{2a^2x}+\frac{55b^2}{3a^3}+\frac{25b^3x}{a^4}+\frac{10b^4x^2}{a^5}\right)\frac{1}{\mathrm{X}^3}$$

$$-\frac{10b^2}{a^6}\log.\frac{\mathrm{X}}{x}.$$

Ex. 13. Let $udx = \dfrac{dx}{(a+bx+cx^2)^n}$,

$$a + bx + cx^2 = \mathrm{X},\ 4ac - b^2 = k,$$

$$\int\frac{dx}{\mathrm{X}}=\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{dx}{\mathrm{X}^2}=\frac{2cx+b}{k\mathrm{X}}+\frac{2c}{k}\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{dx}{\mathrm{X}^3}=\left(\frac{1}{2k\mathrm{X}^2}+\frac{3c}{k^2\mathrm{X}}\right)(2cx+b)+\frac{6c^2}{k^2}\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{dx}{\mathrm{X}^4}=\left(\frac{1}{3k\mathrm{X}^3}+\frac{5c}{3k^2\mathrm{X}^2}+\frac{10c^2}{k^3\mathrm{X}}\right)(2cx+b)+\frac{20c^3}{k^3}\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{dx}{\mathrm{X}^5}=\left(\frac{1}{4k\mathrm{X}^4}+\frac{7c}{6k^2\mathrm{X}^3}+\frac{35c^2}{6k^3\mathrm{X}^2}+\frac{35c^3}{k^4\mathrm{X}}\right)(2cx+b)+\frac{70c^4}{k^4\mathrm{X}}\int\frac{dx}{\mathrm{X}}$$

When x retains its signification in these examples, we have in general

$$\int\frac{dx}{\mathrm{x}}=\frac{2}{\sqrt{(4ac-b^2)}}\ \text{tang.}^{-1}\frac{2cx+b}{\sqrt{(4ac-b^2)}}$$

$$=\frac{1}{\sqrt{(b^2-4ac)}}\log.\frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}.$$

The first form is real when $4ac - b^2$ is positive; the second is so when $4ac - b^2$ is negative. Hence there arises

I. If $4ac - b^2$ be *positive* $(4ac - b^2 = k)$.

$$\int\frac{dx}{\mathrm{x}}=\frac{2}{\sqrt{k}}\ \text{tang.}^{-1}\frac{2cx+b}{\sqrt{k}}=\frac{2}{\sqrt{k}}\ \text{cot.}^{-1}\frac{\sqrt{k}}{2cx+b}$$

$$=\frac{2}{\sqrt{k}}\ \text{sec.}^{-1}\frac{2\sqrt{c\mathrm{x}}}{\sqrt{k}},$$

$$=\frac{2}{\sqrt{k}}\ \text{cosec.}^{-1}\frac{2\sqrt{c\mathrm{x}}}{2cx+b}=\frac{2}{\sqrt{k}}\ \text{cos.}^{-1}\frac{\sqrt{k}}{2\sqrt{c\mathrm{x}}}$$

$$=\frac{2}{\sqrt{k}}\ \text{sin.}^{-1}\frac{2cx+b}{2\sqrt{c\mathrm{x}}},$$

$$=\frac{1}{\sqrt{k}}\ \text{sin.}^{-1}\frac{(2cx+b)\sqrt{k}}{2c\mathrm{x}}=\frac{1}{\sqrt{k}}\ \text{cos.}^{-1}\left(\frac{k}{2c\mathrm{x}}-1\right),$$

$$=\frac{1}{\sqrt{k}}\ \text{ver. sin.}^{-1}\frac{(2cx+b)^2}{2c\mathrm{x}}.$$

And when $\int\frac{dx}{\mathrm{x}}$ vanishes by putting $x = 0$,

$$\int\frac{dx}{\mathrm{x}}=\frac{2}{\sqrt{k}}\ \text{tang.}^{-1}\frac{x\sqrt{k}}{2a+bx}=\frac{2}{\sqrt{k}}\ \text{cot.}^{-1}\frac{2a+bx}{x\sqrt{k}}$$

$$=\frac{2}{\sqrt{k}}\ \text{sec.}^{-1}\frac{2\sqrt{a\mathrm{x}}}{2a+bx},$$

$$=\frac{2}{\sqrt{k}}\ \text{cosec.}^{-1}\frac{2\sqrt{a\mathrm{x}}}{x\sqrt{k}}=\frac{2}{\sqrt{k}}\ \text{sin.}^{-1}\frac{x\sqrt{k}}{2\sqrt{a\mathrm{x}}}$$

$$=\frac{2}{\sqrt{k}}\ \text{cos.}^{-1}\frac{2a+bx}{2\sqrt{a\mathrm{x}}},$$

$$=\frac{1}{\sqrt{k}}\ \text{sin.}^{-1}\frac{(2ax+bx^2)\sqrt{k}}{2a\mathrm{x}}=\frac{1}{\sqrt{k}}\ \text{ver. sin.}^{-1}\frac{kx^2}{2a\mathrm{x}}.$$

II. If $4ac - b^2$ be *negative* $(b^2 - 4ac = k')$.

$$\int\frac{dx}{\mathrm{X}}=\frac{1}{\sqrt{k'}}\log.\frac{2cx+b-\sqrt{k'}}{2cx+b+\sqrt{k'}}=\frac{2}{\sqrt{k'}}\log.\frac{2cx+b-\sqrt{k'}}{2\sqrt{c\mathrm{X}}},$$

and when the integral vanishes by putting $x=0$,

$$\int\frac{dx}{\mathrm{X}}=\frac{1}{\sqrt{k'}}\log.\frac{(b+\sqrt{k'})(2cx+b-\sqrt{k'})}{(b-\sqrt{k'})(2cx+b+\sqrt{k'})}.$$

In both kinds of integrals, $\sqrt{k}$ and $\sqrt{k'}$ may be taken either positive or negative.

Ex. 14. Let $udx=\dfrac{x^m dx}{a+bx+cx^2}$,

$$a+bx+cx^2=\mathrm{X},$$

$$\int\frac{dx}{\mathrm{X}}=\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{xdx}{\mathrm{X}}=\frac{1}{2c}\log.\mathrm{X}-\frac{b}{2c}\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{x^2dx}{\mathrm{X}}=\frac{x}{c}-\frac{b}{2c^2}\log.\mathrm{X}+\left(\frac{b^2}{2c^2}-\frac{a}{c}\right)\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{x^3dx}{\mathrm{X}}=\frac{x^2}{2c}-\frac{bx}{c^2}+\left(\frac{b^2}{2c^3}-\frac{a}{2c^2}\right)\log.\mathrm{X}-\left(\frac{b^3}{2c^3}-\frac{3ab}{2c^2}\right)\int\frac{dx}{\mathrm{X}}.$$

Ex. 15. Let $udx=\dfrac{x^m dx}{(a+bx+cx^2)^3}$,

$$a+bx+cx^2=\mathrm{X},\ 4ac-b^2=k,$$

$$\int\frac{dx}{\mathrm{X}^3}=\left(\frac{1}{2k\mathrm{X}^2}+\frac{3c}{k^2\mathrm{X}}\right)(2cx+b)+\frac{6c^2}{k^2}\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{xdx}{\mathrm{X}^3}=-\frac{1}{4c\mathrm{X}^2}-\frac{b}{2c}\int\frac{dx}{\mathrm{X}^3},$$

$$\int\frac{x^2dx}{\mathrm{X}^3}=\left(-\frac{x}{3c}+\frac{b}{12c^2}\right)\frac{1}{\mathrm{X}^2}+\left(\frac{b^2}{6c^2}+\frac{a}{3c}\right)\int\frac{dx}{\mathrm{X}^3},$$

$$\int\frac{x^3dx}{\mathrm{X}^3}=\left(-\frac{x^2}{2c}-\frac{a}{4c^2}\right)\frac{1}{\mathrm{X}^2}-\frac{ab}{2c^2}\int\frac{dx}{\mathrm{X}^3},$$

$$\int\frac{x^4dx}{\mathrm{X}^3}=\left(-\frac{x^3}{c}-\frac{bx^2}{2c^2}-\frac{ax}{c^2}\right)\frac{1}{\mathrm{X}^2}+\frac{a^2}{c^2}\int\frac{dx}{\mathrm{X}^3}.$$

Ex. 16. Let $udx=\dfrac{dx}{x^m(a+bx+cx^2)^4}$;

$$a+bx+cx^2=\mathrm{X},$$

$$\int\frac{dx}{x\mathrm{X}^4}=\frac{1}{6a\mathrm{X}^3}+\frac{1}{4a^2\mathrm{X}^2}+\frac{1}{2a^3\mathrm{X}}+\frac{1}{2a^4}\log.\frac{x^2}{\mathrm{X}}-\frac{b}{2a}\int\frac{dx}{\mathrm{X}^4}$$
$$-\frac{b}{2a^2}\int\frac{dx}{\mathrm{X}^3}-\frac{b}{2a^3}\int\frac{dx}{\mathrm{X}^2}-\frac{b}{2a^4}\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{dx}{x^2\mathrm{X}^4}=-\frac{1}{ax\mathrm{X}^3}-\frac{4b}{a}\int\frac{dx}{x\mathrm{X}^4}-\frac{7c}{a}\int\frac{dx}{\mathrm{X}^4},$$

$$\int\frac{dx}{x^3\mathrm{X}^4}=\left(-\frac{1}{2ax^2}+\frac{5b}{2ax^2}\right)\frac{1}{\mathrm{X}^3}+\left(\frac{10b^2}{a^2}-\frac{4c}{a}\right)\int\frac{dx}{x\mathrm{X}^4}$$
$$+\frac{35bc}{2a^2}\int\frac{dx}{\mathrm{X}^4},$$

$$\int\frac{dx}{x^4\mathrm{X}^4}=\left[-\frac{1}{3ax^3}+\frac{b}{a^2x^2}-\left(\frac{5b^2}{a^3}-\frac{3c}{a^2}\right)\frac{1}{x}\right]\frac{1}{\mathrm{X}^3}$$
$$-\left(\frac{20b^3}{a^3}-\frac{20bc}{a^2}\right)\int\frac{dx}{x\mathrm{X}^4}-\left(\frac{35b^2c}{a^3}-\frac{21c^2}{a^2}\right)\int\frac{dx}{\mathrm{X}^4},$$

$$\int\frac{dx}{x^5\mathrm{X}^4}=-\frac{1}{4ax^4\mathrm{X}^3}-\frac{7b}{4a}\int\frac{dx}{x^4\mathrm{X}^4}-\frac{5c}{2a}\int\frac{dx}{x^3\mathrm{X}^4}.$$

Ex. 17. Let $udx=\dfrac{dx}{x^m(a+bx+cx^2)^5}$,

$$a+bx+cx^2=\mathrm{X},$$

$$\int\frac{dx}{x\mathrm{X}^5}=\frac{1}{8a\mathrm{X}^4}+\frac{1}{6a^2\mathrm{X}^3}+\frac{1}{4a^3\mathrm{X}^2}+\frac{1}{2a^4\mathrm{X}}+\frac{1}{2a^5}\log.\frac{x^2}{\mathrm{X}}-\frac{b}{2a}\int\frac{dx}{\mathrm{X}^5}$$
$$-\frac{b}{2a^2}\int\frac{dx}{\mathrm{X}^4}-\frac{b}{2a^3}\int\frac{dx}{\mathrm{X}^3}-\frac{b}{2a^4}\int\frac{dx}{\mathrm{X}^2}-\frac{b}{2a^5}\int\frac{dx}{\mathrm{X}},$$

$$\int\frac{dx}{x^2\mathrm{X}^5}=-\frac{1}{ax\mathrm{X}^4}-\frac{5b}{a}\int\frac{dx}{x\mathrm{X}^5}-\frac{9c}{a}\int\frac{dx}{\mathrm{X}^5},$$

$$\int\frac{dx}{x^3\mathrm{X}^5}=\left(-\frac{1}{2ax^2}+\frac{3b}{a^2x}\right)\frac{1}{\mathrm{X}^4}+\left(\frac{15b^2}{a^2}-\frac{5c}{a}\right)\int\frac{dx}{x\mathrm{X}^5}+\frac{27bc}{a^2}\int\frac{dx}{\mathrm{X}^5}$$

$$\int\frac{dx}{x^4\mathrm{X}^5}=\left[-\frac{1}{3ax^3}+\frac{7b}{6a^2x^2}-\left(\frac{7b^2}{a^3}-\frac{11c}{3a^2}\right)\frac{1}{x}\right]\frac{1}{\mathrm{X}^4}$$
$$-\left(\frac{35b^3}{a^3}-\frac{30bc}{a^2}\right)\int\frac{dx}{x\mathrm{X}^5}-\left(\frac{63b^2c}{a^3}-\frac{33c^2}{a^2}\right)\int\frac{dx}{\mathrm{X}^5},$$

$$\int\frac{dx}{x^5\mathrm{X}^5}=-\frac{1}{4ax^4\mathrm{X}^4}-\frac{2b}{a}\int\frac{dx}{x^4\mathrm{X}^5}-\frac{3c}{a}\int\frac{dx}{x^3\mathrm{X}^5}.$$

Ex. 18. Let $udx = \dfrac{x^m dx}{(a+bx^4)^2}$,

$$a + bx^4 = \mathrm{x},$$

$$\int\frac{dx}{\mathrm{x}^2}=\frac{x}{4a\mathrm{x}}+\frac{3}{4a}\int\frac{dx}{\mathrm{x}},$$

$$\int\frac{xdx}{\mathrm{x}^2}=\frac{x^2}{4a\mathrm{x}}+\frac{1}{2a}\int\frac{xdx}{\mathrm{x}},$$

$$\int\frac{x^2dx}{\mathrm{x}^2}=\frac{x^3}{4a\mathrm{x}}+\frac{1}{4a}\int\frac{x^2dx}{\mathrm{x}},$$

$$\int\frac{x^3dx}{\mathrm{x}^2}=-\frac{1}{4b\mathrm{x}},$$

$$\int\frac{x^4dx}{\mathrm{x}^2}=-\frac{x}{4b\mathrm{x}}+\frac{1}{4b}\int\frac{dx}{\mathrm{x}},$$

$$\int\frac{x^5dx}{\mathrm{x}^2}=-\frac{x^2}{4b\mathrm{x}}+\frac{1}{2b}\int\frac{xdx}{\mathrm{x}},$$

$$\int\frac{x^6dx}{\mathrm{x}^2}=-\frac{x^3}{4b\mathrm{x}}+\frac{3}{4b}\int\frac{x^2dx}{\mathrm{x}}.$$

Ex. 19. Let $udx = \dfrac{x^m dx}{\mathrm{x}^3}$, $a + bx^4 = \mathrm{x}$,

$$\int\frac{dx}{\mathrm{x}^3}=\left(\frac{7bx^4}{32a^2}+\frac{11x}{32a}\right)\frac{1}{\mathrm{x}^2}+\frac{21}{32a^2}\int\frac{dx}{\mathrm{x}},$$

$$\int\frac{xdx}{\mathrm{x}^3}=\left(\frac{3bx^6}{16a^2}+\frac{5x^2}{16a}\right)\frac{1}{\mathrm{x}^2}+\frac{3}{8a^2}\int\frac{xdx}{\mathrm{x}},$$

$$\int\frac{x^2dx}{\mathrm{x}^3}=\left(\frac{5bx^7}{32a^2}+\frac{9x^3}{32a}\right)\frac{1}{\mathrm{x}^2}+\frac{5}{32a^2}\int\frac{x^2dx}{\mathrm{x}},$$

$$\int\frac{x^3dx}{\mathrm{x}^3}=-\frac{1}{8b\mathrm{x}^2},$$

$$\int\frac{x^4dx}{\mathrm{x}^3}=\left(\frac{x^5}{32a}-\frac{3x}{32b}\right)\frac{1}{\mathrm{x}^2}+\frac{3}{32ab}\int\frac{dx}{\mathrm{x}},$$

$$\int\frac{x^5dx}{\mathrm{x}^3}=\left(\frac{x^6}{16a}-\frac{x^2}{16b}\right)\frac{1}{\mathrm{x}^2}+\frac{1}{8ab}\int\frac{xdx}{\mathrm{x}},$$

$$\int\frac{x^6dx}{\mathrm{x}^3}=\left(\frac{3x^7}{32a}-\frac{x^3}{32ab}\right)\frac{1}{\mathrm{x}^2}+\frac{3}{32ab}\int\frac{x^2dx}{\mathrm{x}}$$

Ex. 20. Let $udx = \dfrac{dx}{x^m(a+bx^4)^2}$,

$$a + bx^4 = \text{x},$$

$$\int \frac{dx}{x\text{x}} = \frac{\log. x}{a} - \frac{\log. \text{x}}{4a} = \frac{1}{4a}\log. \frac{x^4}{\text{x}} = -\frac{1}{4a}\log. \frac{\text{x}}{x^4},$$

$$\int \frac{dx}{x^2\text{x}} = -\frac{1}{ax} - \frac{b}{a}\int \frac{x^2dx}{\text{x}},$$

$$\int \frac{dx}{x^3\text{x}} = -\frac{1}{2ax^2} - \frac{b}{a}\int \frac{xdx}{\text{x}},$$

$$\int \frac{dx}{x^4\text{x}} = -\frac{1}{3ax^3} - \frac{b}{a}\int \frac{dx}{\text{x}},$$

$$\int \frac{dx}{x^5\text{x}} = -\frac{1}{4ax^4} - \frac{b}{a}\int \frac{dx}{x\text{x}},$$

$$\int \frac{dx}{x^6\text{x}} = -\frac{1}{5ax^5} + \frac{b}{a^2x} + \frac{b^2}{a^2}\int \frac{x^2dx}{\text{x}},$$

$$\int \frac{dx}{x^7\text{x}} = -\frac{1}{6ax^6} + \frac{b}{2a^2x^2} + \frac{b^2}{a^2}\int \frac{xdx}{\text{x}}.$$

Ex. 21. Let $udx = \dfrac{dx}{x^m(a+bx^4)^2}$, $a + bx^4 = \text{x}$,

$$\int \frac{dx}{x\text{x}^2} = \frac{1}{4a\text{x}} + \frac{1}{a}\int \frac{dx}{x\text{x}},$$

$$\int \frac{dx}{x^2\text{x}^2} = \left(-\frac{1}{ax} - \frac{5bx^3}{4a^2}\right)\frac{1}{\text{x}} - \frac{5b}{4a^2}\int \frac{x^2dx}{\text{x}},$$

$$\int \frac{dx}{x^3\text{x}^2} = \left(-\frac{1}{2ax^2} - \frac{3bx^2}{4a^2}\right)\frac{1}{\text{x}} - \frac{3b}{2a^2}\int \frac{xdx}{\text{x}},$$

$$\int \frac{dx}{x^4\text{x}^2} = \left(-\frac{1}{3ax^3} - \frac{7bx}{12a^2}\right)\frac{1}{\text{x}} - \frac{7b}{4a^2}\int \frac{dx}{\text{x}},$$

$$\int \frac{dx}{x^5\text{x}^2} = \left(-\frac{1}{4ax^4} - \frac{b}{2a^2}\right)\frac{1}{\text{x}} - \frac{2b}{a^2}\int \frac{dx}{x\text{x}},$$

$$\int \frac{dx}{x^6\text{x}^2} = \left(-\frac{1}{5ax^5} + \frac{9b}{5a^2x} + \frac{9b^2x^3}{4a^3}\right)\frac{1}{\text{x}} + \frac{9b^2}{4a^3}\int \frac{x^2dx}{\text{x}},$$

$$\int \frac{dx}{x^7\text{x}^2} = \left(-\frac{1}{6ax^6} + \frac{5b}{6a^2x^2} + \frac{5b^2x^2}{4a^3}\right)\frac{1}{\text{x}} + \frac{5b^2}{2a^3}\int \frac{xdx}{\text{x}}.$$

SECTION IV.

Of the integration of differentials, of which the coefficients are irrational.

(215.) The integration of differentials, of which the coefficients are irrational functions of the variable, is, in general, effected by a transformation, by which the function is *rationalised*. Such transformations must be suggested by the expertness and address of the analyst rather than by any general rules. Our knowledge in this part of the integral calculus is considerably limited, and there are numerous classes of differentials, the integrals of which have never yet been assigned under a finite form. In the present section we shall attempt to reduce to a few comprehensive classes the principal irrational differentials which have been integrated in finite terms.

I. The first class includes the elementary differentials

$$\pm\frac{dx}{\sqrt{1-x^2}},\quad \pm\frac{dx}{x\sqrt{x^2-1}},\quad \pm\frac{dx}{\sqrt{2x-x^2}},$$

of which the integrals have been assigned in (198.).

II. All differentials, whose coefficients are of the form

$$\text{F}(x, x^a, x^b, x^c, \cdots\cdot)$$

the functional sign F denoting a rational function; but $a, b, c, \cdots\cdot$ being any *fractions*.

III. All differentials, whose coefficients are of the form

$$\text{F}(x, \text{X}^a, \text{X}^b, \text{X}^c, \cdots\cdot).$$

Where F denotes as before a rational function, and X is a function of x of the form $\text{A}+\text{B}x$, and the exponents are any fractions.

IV. All differentials, whose coefficients come under the preceding form, X denoting a function of x of the form

$$\frac{A+Bx}{A'+B'x}.$$

V. All differentials, whose coefficients come under the form

$$F[x, (A + Bx + Cx^2)^{\frac{1}{2}}].$$

VI. Differentials, whose coefficients have the form

$$x^{m-1}(A + Bx^n)^k,$$

k being a fraction. These are called *binomial differentials.*

VII. Differentials, whose coefficients are of the form

$$F(x^{mn}, X^a, X^b, X^c, \cdots) \times x^{n-1}.$$

Where $X = A + Bx^n$, and $a, b, c, \cdots$ are any fractions.

VIII. Differentials, whose coefficients are of the preceding form, X denoting a function of the form

$$\frac{A+Bx^n}{A'+B'x^n}.$$

IX. Differentials, whose coefficients are of the form

$$x^m \times F[x^n, (A + Bx^n + Cx^{2n})^{\frac{1}{2}}].$$

In all these classes the functional sign F denotes a rational function of the quantities within the parenthesis which follows it. We shall now proceed to explain the methods of integration used in these cases successively.

(216.) I. The first class needs no further observation, as the form of the integrals are immediately determined by the differential calculus. (See 198.).

(217.) II. The differentials of this class are of the form

$$F(x, x^a, x^b, x^c, \cdots)dx.$$

They may be rationalised by reducing the fractions $a, b, c, \cdots$ to a common denominator. Let this be D, and let

$$x^{\frac{1}{D}} = z, \quad \therefore x = z^D, \quad \therefore dx = Dz^{D-1}dz.$$

It is evident also, that x^a, x^b, x^c, ... are integral powers of z. These transformations reduce the differential to the form

$$\text{F}\{z^{\text{D}}, z^{a'}, z^{b'}, \dots \}\text{D}z^{\text{D}-1}dz,$$

where D, a', b', c', ... are integers. This being rational, may be integrated by the rules in Section II.

(218.) III. The differentials of this class are of the form

$$\text{F}(x, \text{X}^a, \text{X}^b, \text{X}^c, \dots)dx.$$

This class may be reduced to the preceding, thus,

$$\text{X} - \text{A} = \text{B}x, \therefore x = \frac{\text{X}-\text{A}}{\text{B}},$$

$$\therefore dx = \frac{d\text{X}}{\text{B}}.$$

Hence the differential becomes

$$\frac{1}{\text{B}}\text{F}\left\{\frac{\text{X}-\text{A}}{\text{B}}, \text{X}^a, \text{X}^b, \dots \right\}d\text{X},$$

$$\text{or } \frac{1}{\text{B}}\text{F}\left\{\text{X}, \text{X}^a, \text{X}^b, \dots \right\}d\text{X},$$

which is included in class II.

(219.) IV. This class may also be reduced to II. For

$$\text{X} = \frac{\text{A}+\text{B}x}{\text{A}'+\text{B}'x}, \therefore x = -\frac{\text{A}-\text{A}'\text{X}}{\text{B}-\text{B}'\text{X}},$$

$$dx = \frac{\text{BA}'-\text{B}'\text{A}}{(\text{B}-\text{B}'\text{X})^2}d\text{X}.$$

By these substitutions, the differential assumes the form

$$\text{F}(\text{X}, \text{X}^a, \text{X}^b, \dots)d\text{X},$$

which comes under class II.

(220.) V. This class of differentials is not rationalised with the same facility as the former. It will be necessary to consider two cases, where $\text{C} > 0$ or < 0. If $\text{C} = 0$, the differential comes under class III.

1°. If $\text{C} > 0$, let

$$\text{A} + \text{B}x + \text{C}x^2 = \text{C}(x + y)^2,$$

$$\therefore x = \frac{A - Cy^2}{2Cy - B}, \quad dx = -\frac{2C(A - By + Cy^2)}{(B - 2Cy)^2} dy,$$

$$\therefore \sqrt{A + Bx + Cx^2} = \sqrt{C} \cdot \frac{A - By + Cy^2}{2Cy - B}.$$

By which substitutions, the differential becomes rational.

2°. If $C < 0$, let x', x'', be the roots of the equation

$$A + Bx - Cx^2 = 0.$$

Hence

$$A + Bx - Cx^2 = -C(x - x')(x - x'').$$

Let

$$\sqrt{C(x - x')(x'' - x)} = (x - x')Cy,$$

$$\therefore x = \frac{Cx'y^2 + x''}{Cy^2 + 1}, \quad dx = \frac{2(x' - x'')Cy}{(Cy^2 + 1)^2} dy,$$

$$\therefore \sqrt{A + Bx - Cx^2} = \frac{(x'' - x')Cy}{Cy^2 + 1}.$$

It is obvious, since $C < 0$, that the roots x', x'', are real.

Under this class are comprehended differentials of the forms

$$F(x, \sqrt{A + Cx^2})dx,$$

$$F(x, \sqrt{Bx + Cx^2})dx.$$

The former is the case where $B = 0$, and the latter where $A = 0$.

(221.) VI. This class of differentials cannot be always rationalised by any known methods. In some cases, however, this can be effected. It will not render the results less general to consider the exponents m and n integers, and $n > 0$. For if they were fractional, let D be their common denominator. After the transformation, effected by substituting z^D for x, the exponents would become integral; and in like manner, if n were negative, by substituting $\frac{1}{z}$ for x, the exponent of z under the radical would become > 0.

If then m and n be considered as integers, and $n > 0$, the formula may be rationalised whenever either $\frac{m}{n}$ or $\frac{m}{n} + k$ is an integer, whether positive or negative. Since k is a fraction, let it $= \frac{p}{q}$, p and q being integers.

1°. If $\frac{m}{n}$ be an integer, let $A + Bx^n = y^q$, $\because$

$$(A + Bx^n)^{\frac{p}{q}} = y^p,$$

$$\because x^n = \frac{y^q - A}{B}, \quad x^m = \left(\frac{y^q - A}{B}\right)^{\frac{m}{n}},$$

$$x^{m-1}dx = \frac{qy^{q-1}}{nB}\left(\frac{y^q - A}{B}\right)^{\frac{m-n}{n}} dy.$$

By these substitutions, the differential becomes

$$\frac{q}{nB} \cdot y^{p+q-1}\left(\frac{y^q - A}{B}\right)^{\frac{m}{n}-1} dy,$$

which is rational, since $\frac{m}{n}$ is an integer.

2°. If $\frac{m}{n} + \frac{p}{q}$ be an integer, let $A + Bx^n = x^n y^q$, $\because$

$$x^n = \frac{A}{y^q - B},$$

$$A + Bx^n = \frac{Ay^q}{y^q - B},$$

$$(A + Bx^n)^{\frac{p}{q}} = \frac{A^{\frac{p}{q}} y^p}{(y^q - B)^{\frac{p}{q}}},$$

$$x^m = \frac{A^{\frac{m}{n}}}{(y^q - B)^{\frac{m}{n}}},$$

$$x^{m-1}dx = -\frac{q\text{A}^{\frac{m}{n}}y^{q-1}}{n(y^q-\text{B})^{\frac{m}{n}+1}}dy.$$

By these substitutions, the proposed differential becomes

$$\frac{q}{n}\text{A}^{\frac{m}{n}+\frac{p}{q}}\frac{y^{p+q-1}}{(y^q-\text{B})^{\frac{m}{n}+\frac{p}{q}+1}}dy,$$

which is rational, since $\frac{m}{n}+\frac{p}{q}$ is an integer.

(222.) These are the only cases in which methods of rationalising binomial differentials have yet been assigned, and are therefore the only cases where their integrals can be obtained by the methods given in Sect. II. Integration by parts, however, furnishes means of reducing the integration of given binomial differentials to that of other binomial differentials with lower exponents; in which case the final integration may frequently be completed by analytical artifice. In general, then, the integration of the formula

$$x^{m-1}\text{X}^k dx$$

may be made to depend on the integration of a similar formula in which the exponent of either x or X is less than $m-1$ or k. It will be necessary to consider separately the cases where the m and k are positive and negative. We shall therefore establish the following equations:

I. If $m > 0$.

$$\int x^{m-1}\text{X}^k dx = \frac{x^{m-n}\text{X}^{k+1}}{(kn+m)\text{B}} - \frac{(m-n)\text{A}}{(kn+m)\text{B}}\int x^{m-n-1}\text{X}^k dx,$$

in which $m-1$, $m-2$, $m-3$, ... being successively substituted for m, the exponent of x will be continually reduced.

II. If $k > 0$.

$$\int x^{m-1}\text{X}^k dx = \frac{x^m\text{X}^k}{kn+m} + \frac{kn\text{A}}{kn+m}\int x^{m-1}\text{X}^{k-1}dx,$$

in which $k-1$, $k-2$, &c. being successively substituted for k, the exponent of X is continually reduced.

III. If $m < 0$.

$$\int x^{-m-1}\mathrm{X}^k dx = -\frac{x^{-m}\mathrm{X}^{k+1}}{m\mathrm{A}} - \frac{(m-n-kn)\mathrm{B}}{m\mathrm{A}}\int x^{-m+n-1}\mathrm{X}^k dx,$$

where the negative exponent of x is diminished.

IV. If $k < 0$.

$$\int x^{m-1}\mathrm{X}^{-k}dx = \frac{x^m\mathrm{X}^{-k+1}}{(k-1)n\mathrm{A}} - \frac{m+n-kn}{(k-1)n\mathrm{A}}\int x^{m-1}\mathrm{X}^{-k+1}dx.$$

We shall consider these formulæ successively.

(223.) 1°. Let

$$\int x^{m-1}\mathrm{X}^k dx = \int \mathrm{X}' d\mathrm{X}'' = \mathrm{X}'\mathrm{X}'' - \int \mathrm{X}'' d\mathrm{X}',$$

where $\mathrm{X} = \mathrm{A} + \mathrm{B}x^n$.

The formula $\int x^{m-1}\mathrm{X}^k dx$, may be put under the form $\int x^{m-n}\mathrm{X}^k x^{n-1}dx$, so that we may suppose

$$\mathrm{X}' = x^{m-n}, \quad d\mathrm{X}'' = \mathrm{X}^k x^{n-1}dx,$$

$$\therefore d\mathrm{X}' = (m-n)x^{m-n-1}dx, \quad \mathrm{X}'' = \frac{\mathrm{X}^{k+1}}{(k+1)n\mathrm{B}},$$

since $d\mathrm{X} = n\mathrm{B}x^{n-1}dx$. Hence we find

$$\int x^{m-1}\mathrm{X}^k dx = \frac{x^{m-n}\mathrm{X}^{k+1}}{(k+1)n\mathrm{B}} - \frac{m-n}{(k+1)n\mathrm{B}}\int x^{m-n-1}\mathrm{X}^{k+1}dx.$$

But

$$\mathrm{X}^{k+1} = \mathrm{X}^k\mathrm{X} = \mathrm{X}^k(\mathrm{A} + \mathrm{B}x^n),$$

$$\therefore \mathrm{X}^{k+1} = \mathrm{A}\mathrm{X}^k + \mathrm{B}x^n\mathrm{X}^k.$$

Hence

$$\int x^{m-n-1}\mathrm{X}^{k+1}dx = \mathrm{A}\int x^{m-n-1}\mathrm{X}^k dx + \mathrm{B}\int x^{m-1}\mathrm{X}^k dx.$$

Making this substitution, and collecting the integrals

$$\left(1 + \frac{m-n}{(k+1)n}\right)\int x^{m-1}\mathrm{X}^k dx =$$

$$\frac{x^{m-n}\mathrm{X}^{k+1} - \mathrm{A}(m-n)\int x^{m-n-1}\mathrm{X}^k dx}{(k+1)n\mathrm{B}},$$

$$\therefore \int x^{m-1}\mathrm{X}^k dx = \frac{x^{m-n}\mathrm{X}^{k+1}}{(kn+m)\mathrm{B}} - \frac{(m-n)\mathrm{A}}{(kn+m)\mathrm{B}}\int x^{m-n-1}\mathrm{X}^k dx \cdots [1].$$

Thus the integration of the given differential is made to depend on that of the differential

$$x^{m-n-1}\mathrm{X}^k dx.$$

It is obvious that by a similar process the integration of this last may be made to depend on that of

$$x^{m-2n-1}\mathrm{X}^k dx\,;$$

and by continuing the process, the exponent $m-1$ will be successively reduced each step by n. If m be a multiple of n, the integration will by this process be completely effected, for the coefficient of the integral after each step is m diminished by a multiple of n, which must therefore ultimately vanish, and the integration will be completed. This is the case already mentioned, where $\frac{m}{n}$ is an integer.

This formula of reduction expressed in general is

$$\int x^{m-(r-1)n-1}\mathrm{X}^k dx = \frac{x^{m-rn}\,\mathrm{X}^{k+1}}{[kn+m-(r-1)n]\mathrm{B}}$$

$$-\frac{(m-rn)\mathrm{A}}{[kn+m-(r-1)n]\mathrm{B}}\int x^{m-rn-1}\mathrm{X}^k dx,$$

where r is the number of reductions which have been made.

(224.) 2°. We may also, without difficulty, obtain the formula for the reduction of this integral, by which the exponent of X is continually diminished. This may be thus effected:

$$\mathrm{X}^k = \mathrm{X}^{k-1}\mathrm{X} = \mathrm{A}\mathrm{X}^{k-1} + \mathrm{B}\mathrm{X}^{k-1}x^n,$$

$$\therefore \int x^{m-1}\mathrm{X}^k dx = \mathrm{A}\int x^{m-1}\mathrm{X}^{k-1}dx + \mathrm{B}\int x^{m+n-1}\mathrm{X}^{k-1}dx.$$

But by [1], we find

$$\int x^{m+n-1}\mathrm{X}^{k-1}dx = \frac{x^m\mathrm{X}^k}{(kn+m)\mathrm{B}}$$

$$-\frac{m\mathrm{A}}{(kn+m)\mathrm{B}}\int x^{m-1}\mathrm{X}^{k-1}dx.$$

Substituting this in the preceding equation, we find

$$\int x^{m-1}\mathrm{X}^k dx = \frac{x^m\mathrm{X}^k}{kn+m} + \frac{kn\mathrm{A}}{kn+m}\int x^{m-1}\mathrm{X}^{k-1}dx \cdots\cdot [2].$$

By the successive application of this principle, the ex-

ponent of x may be diminished at each step of the process by unity.

(225.) 3°. If m or k be negative, the formulæ [1] or [2] will not effect the required reduction, for the exponents will in that case continually increase. They will, however, by a slight change, give formulæ fitted for the purpose.

By [1], we find

$$\int x^{m-n-1}\mathrm{X}^k dx = \frac{x^{m-n}\mathrm{X}^{k+1}}{(m-n)\mathrm{A}} - \frac{(kn+m)\mathrm{B}}{(m-n)\mathrm{A}}\int x^{m-1}\mathrm{X}^k dx.$$

Substituting for m in this $n - m$, we find

$$\int x^{-m-1}\mathrm{X}^k dx = -\frac{x^{-m}\mathrm{X}^{k+1}}{m\mathrm{A}} - \frac{(m-n-kn)\mathrm{B}}{m\mathrm{A}}\int x^{-m+n-1}\mathrm{X}^k dx \cdots \quad [3].$$

By the successive application of this formula, the exponent $-m-1$ is continually diminished.

(226.) 4°. Also from [2], we find

$$\int x^{m-1}\mathrm{X}^{k-1} dx = -\frac{x^m\mathrm{X}^k}{kn\mathrm{A}} + \frac{kn+m}{kn\mathrm{A}}\int x^{m-1}\mathrm{X}^k dx.$$

Substituting $1-k$ for k, we find

$$\int x^{m-1}\mathrm{X}^{-k} dx = \frac{x^m\mathrm{X}^{-k+1}}{(k-1)n\mathrm{A}} - \frac{m+n-kn}{(k-1)n\mathrm{A}}\int x^{m-1}\mathrm{X}^{-k+1} dx \cdots\cdots [4],$$

by which the exponent k is continually diminished.

(227.) V. The integration of

$$\mathrm{F}(x^{mn}, \mathrm{X}^a, \mathrm{X}^b, \cdot\cdot\cdot\cdot) x^{n-1} dx,$$

where $\mathrm{X} = \mathrm{A} + \mathrm{B}x^n$ is effected by the transformation

$$\mathrm{X} = z^{\mathrm{D}},$$

D being the common denominator of all the fractional exponents $a, b, c, \cdot\cdot\cdot\cdot$ For then

$$x^n = \frac{z^{\mathrm{D}} - \mathrm{A}}{\mathrm{B}}, \quad x^{n-1}dx = \frac{\mathrm{D}z^{\mathrm{D}-1}dz}{n\mathrm{B}},$$

by which the proposed formula becomes

$$\mathrm{F}\left[\left(\frac{z^{\mathrm{D}}-\mathrm{A}}{\mathrm{B}}\right)^m, z^{a'}, z^{b'}, \cdot\cdot\cdot\cdot\right]\frac{\mathrm{D}z^{\mathrm{D}-1}dz}{n\mathrm{B}},$$

where $a', b', \dots$ are integers. This is rational with respect to z.

(228.) VI. The same formula when

$$X = \frac{A + Bx^n}{A' + B'x^n}$$

may be rationalised and integrated by the transformation,

$$X = z^D,$$

the result of which will be analogous to the former.

(229.) VII. The differential

$$x^m F(x^n, \sqrt{A + Bx^n + Cx^{2n}})dx$$

may be rationalised by putting $x^n = z$, by which it becomes

$$\frac{1}{n} z^{\frac{m+1}{n} - 1} \cdot F(z, \sqrt{A + Bz + Cz^2})dz,$$

which comes under the form of (V), and may be treated as in (220.).

SECTION V.

Praxis on the integration of differentials, whose coefficients are irrational.

Ex. 1. Let $udx = \dfrac{1 + x^{\frac{1}{2}} - x^{\frac{2}{3}}}{1 + x^{\frac{1}{3}}} dx$. The common denominator of the exponents is 6, $\therefore$ let $x = z^6$,

$$\therefore udx = \frac{6z^5(1 + z^3 - z^4)}{1 + z^2} dz,$$

which, by effecting the division, gives

$$udx = -6\left\{ z^7 dz - z^6 dz - z^5 dz + z^4 dz - z^2 dz + dz - \frac{dz}{1 + z^2} \right\},$$

which being integrated, gives

$$\int udx = -6\left\{\frac{z^8}{8}-\frac{z^7}{7}-\frac{z^6}{6}+\frac{z^5}{5}-\frac{z^3}{3}+z-\tan.^{-1}z\right\}$$

Ex. 2. $udx = \dfrac{dx}{\sqrt{a+bx}}$. Let $z^2=a+bx$, $\because dx=\dfrac{2zdz}{b}$, $\therefore$

$$udx = \frac{2}{b}dz,$$

$$\int udx = \frac{2}{b}\sqrt{a+bx}.$$

Ex. 3. $udx = \dfrac{dx}{x\sqrt{a^2+bx}}$. Let $a^2+bx=z^2$,

$2zdz = bdx$, and $x = \dfrac{z^2-a^2}{b}$. Hence

$$udx = \frac{2dz}{z^2-a^2},$$

which has been integrated in Section III. Ex. 1.

Ex. 4. $udx = \dfrac{dx}{\sqrt{\text{A}+\text{B}x+\text{C}x^2}}$

1°. Let $\text{C}>0$. By the transformation in (220.), this becomes

$$udx = 2\sqrt{\text{C}}\cdot\frac{dy}{\text{B}-2\text{C}y},$$

which being integrated, gives

$$\int udx = -\frac{1}{\sqrt{\text{C}}}l(2\text{C}y-\text{B}) = \frac{1}{\sqrt{\text{C}}}l\frac{1}{2\text{C}y-\text{B}}$$

After substituting for y its value, and concinnating, we find

$$\int udx = \frac{1}{\sqrt{\text{C}}}l\left\{2\text{C}x+\text{B}+2\sqrt{\text{C}}\sqrt{\text{A}+\text{B}x+\text{C}x^2}\right\}.$$

2°. Let $\text{C}<0$. The transformation (220.), gives

$$\int udx = -\int\frac{2dy}{\text{C}y^2+1},$$

$$\because\int udx = \frac{2}{\sqrt{\text{C}}}\cot.^{-1}\sqrt{\text{C}}.y,$$

$$\sqrt{c}\cdot y = \frac{\sqrt{x'' - x}}{\sqrt{x - x'}},$$

$$\therefore \int u dx = \frac{2}{\sqrt{c}} \cot.^{-1} \frac{\sqrt{x'' - x}}{\sqrt{x - x'}}.$$

The integrals just found may be presented under different forms. The following are among the varieties they may assume:

1°. When $c > 0$,

$$\int u dx = \pm \frac{1}{\sqrt{c}}\, l \left\{ 2cx + B \pm 2\sqrt{c}\sqrt{A + Bx + cx^2} \right\} \cdots\cdots [1].$$

When $x = 0$ renders $\int u dx = 0$, then

$$\int u dx = \pm \frac{1}{\sqrt{c}}\, l\, \frac{2cx + B \pm 2\sqrt{c}\sqrt{A + Bx + cx^2}}{B \pm 2\sqrt{AC}} \quad \cdots\cdots [2],$$

the upper signs being taken together, and also the lower.

2°. When $c < 0$,

$$\int u dx = \frac{1}{\sqrt{c}} \sin.^{-1} \frac{2cx - B}{\sqrt{B^2 + 4AC}} \quad \cdots\cdots [3],$$

$$= \frac{1}{\sqrt{c}} \cos.^{-1} \frac{2\sqrt{c}\sqrt{A + Bx - cx^2}}{\sqrt{B^2 + 4AC}} \quad \cdots [4],$$

$$= \frac{1}{\sqrt{c}} \tan.^{-1} \frac{2cx - B}{2\sqrt{c}\sqrt{A + Bx - cx^2}} \quad \cdots [5],$$

$$= \frac{1}{\sqrt{c}} \cot.^{-1} \frac{2\sqrt{c}\sqrt{A + Bx - cx^2}}{2cx - B} \quad \cdots [6],$$

$$= \frac{1}{\sqrt{c}} \sec.^{-1} \frac{\sqrt{B^2 + 4AC}}{2\sqrt{c}\sqrt{A + Bx - cx^2}} \quad \cdots [7],$$

$$= \frac{1}{\sqrt{c}} \operatorname{cosec.}^{-1} \frac{\sqrt{B^2 + 4AC}}{2cx - B} \quad \cdots\cdots [8],$$

$$= \frac{1}{2\sqrt{c}} \text{ver. sin.}^{-1} \frac{2(2cx - B)^2}{B^2 + 4AC} \quad \cdots [9].$$

The constant should be introduced when these are applied to particular cases.

Ex. 5. $udx = \frac{dx}{\sqrt{x^2 \pm a^2}}$. In the preceding example let

C $= 1$, B $= 0$, and A $= \pm a^2$, $\therefore$

$\int udx = l2(x + \sqrt{x^2 \pm a^2}) = l(x + \sqrt{x^2 \pm a^2}) + l2.$

Ex. 6. $udx = \frac{dx}{\sqrt{a^2 - x^2}}$. In Ex. 4. let C $= -1$, B $= 0$,

and A $= a^2$, $\therefore$

$$\int udx = \sin.^{-1}\frac{x}{a},$$

this is one of the elementary integrals in (199.).

Ex. 7. $udx = \sqrt{a^2 + x^2} \cdot dx$. Let $\sqrt{a^2 + x^2} = y - x$,

$\therefore udx = ydx - xdx$, $\therefore \int udx = \int ydx - \frac{1}{2}x^2$.

Substituting for dx its value, and integrating, we find

$$\int ydx = \tfrac{1}{4}y^2 + \tfrac{1}{2}a^2 ly,$$

$$\therefore \int udx = \tfrac{1}{2}x\sqrt{a^2 + x^2} + \tfrac{1}{4}a^2 + \tfrac{1}{2}a^2 l(x + \sqrt{a^2 + x^2}).$$

Ex. 8. $udx = -\frac{dx}{\sqrt{1 - x^2}}$. This is one of the elementary integrals (198.), and

$$\int udx = \cos.^{-1}x = \phi.$$

But it may be also put under the form

$$d\phi\sqrt{-1} = \frac{dx}{\sqrt{x^2 - 1}},$$

$\therefore$ by Ex. 4, putting A $= -1$, B $= 0$, and C $= 1$,

$$\therefore \pm\phi\sqrt{-1} = l[x \pm \sqrt{x^2 - 1}].$$

Since $x = \cos.\phi$, $\sqrt{x^2 - 1} = \sqrt{-1}\sin.\phi$, and since $x = 1$ gives $\phi = 0$, the constant $= 0$. Therefore

$$\pm\phi\sqrt{-1} = l[\cos.\phi \pm \sqrt{-1}\cdot\sin.\phi],$$

$$\therefore e^{\pm\phi\sqrt{-1}} = \cos.\phi \pm \sqrt{-1}\sin.\phi.$$

For the important consequences resulting from this formula, see *Trigonometry*. Also *Diff. Calc.* Sect. III. Exs. 16 and 20.

Ex. 9. Let $udx = \dfrac{x^{m-1}.dx}{\sqrt{1-x^2}}$, m being a positive integer.

By the formula [1] (223.), observing that $k = -\frac{1}{2}$, $A = 1$, $B = -1$, $X = 1 - x^2$, and $n = 2$.

$$\int \frac{x^{m-1}.dx}{\sqrt{1-x^2}} = -\frac{x^{m-2}\sqrt{1-x^2}}{m-1} + \frac{m-2}{m-1}\int\frac{x^{m-3}.dx}{\sqrt{1-x^2}}.$$

Substituting m for $m-1$, we find

$$\int \frac{x^m dx}{\sqrt{1-x^2}} = -\frac{x^{m-1}\sqrt{1-x^2}}{m} + \frac{m-1}{m}\int\frac{x^{m-2}.dx}{\sqrt{1-x^2}}.$$

By successively ascribing to m the values 1, 3, 5 .... we find

$$\int \frac{xdx}{\sqrt{1-x^2}} = -\sqrt{1-x^2},$$

$$\int \frac{x^3dx}{\sqrt{1-x^2}} = -\tfrac{1}{3}x^2\sqrt{1-x^2} + \tfrac{2}{3}\int\frac{xdx}{\sqrt{1-x^2}},$$

$$\int \frac{x^5dx}{\sqrt{1-x^2}} = -\tfrac{1}{5}x^4\sqrt{1-x^2} + \tfrac{4}{5}\int\frac{x^3dx}{\sqrt{1-x^2}},$$

$$\int \frac{x^7dx}{\sqrt{1-x^2}} = -\tfrac{1}{7}x^6\sqrt{1-x^2} + \tfrac{6}{7}\int\frac{x^5dx}{\sqrt{1-x^2}}.$$

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From whence we deduce

$$\int \frac{xdx}{\sqrt{1-x^2}} = -\sqrt{1-x^2},$$

$$\int \frac{x^3dx}{\sqrt{1-x^2}} = -\left(\frac{1}{3}x^2 + \frac{1.2}{1.3}\right)\sqrt{1-x^2},$$

$$\int \frac{x^5dx}{\sqrt{1-x^2}} = -\left(\frac{1}{5}x^4 + \frac{1.4}{3.5}x^2 + \frac{1.2.4}{1.3.5}\right)\sqrt{1-x^2},$$

$$\int \frac{x^7dx}{\sqrt{1-x^2}} = -\left(\frac{1}{7}x^6 + \frac{1.6}{5.7}x^4 + \frac{1.4.6}{3.5.7}x^2 + \frac{1.2.4.6}{1.3.5.7}\right)\sqrt{1-x^2}.$$

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If the values ascribed to m be the numbers of the series 0, 2, 4, 6, $\cdots$ we find, in like manner,

$$^{*}\int\frac{dx}{\sqrt{1-x^2}} = \sin.^{-1}x,$$

$$\int\frac{x^2dx}{\sqrt{1-x^2}} = -\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin.^{-1}x,$$

$$\int\frac{x^4dx}{\sqrt{1-x^2}} = -\left(\frac{1}{4}x^3 + \frac{1.3}{2.4}x\right)\sqrt{1-x^2} + \frac{1.3}{2.4}\sin.^{-1}x,$$

$$\int\frac{x^6dx}{\sqrt{1-x^2}} = -\left(\frac{1}{6}x^5+\frac{1.5}{4.6}x^3+\frac{1.3.5}{2.4.6}x\right)\sqrt{1-x^2}+\frac{1.3.5}{2.4.6}\sin.^{-1}x.$$

. . . .

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Ex. 10. $udx = \dfrac{x^{-m}dx}{\sqrt{1-x^2}}$. This example comes under the general formula [3] (225.). In this case $X^k = (1-x^2)^{-\frac{1}{2}}$, $\therefore$ A $= 1$, B $= -1$, $n = 2$, and $k = -\frac{1}{2}$. Hence

$$\int\frac{x^{-m-1}\cdot dx}{\sqrt{1-x^2}} = -\frac{x^{-m}\sqrt{1-x^2}}{m} + \frac{m-1}{m}\int\frac{x^{-m+1}dx}{\sqrt{1-x^2}}.$$

Substituting $-m$ for $-m-1$, we find

$$\int\frac{dx}{x^m\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{(m-1)x^{m-1}} + \frac{m-2}{m-1}\int\frac{dx}{x^{m-2}\sqrt{1-x^2}}.$$

This formula is subject to an exception when $m = 1$, for then the second member of this equation would be infinite. The integration of this case must therefore be effected upon independent principles. Let $1 - x^2 = z^2$, $\therefore$

$$x = \sqrt{1-z^2},\ dx = -\frac{zdz}{\sqrt{1-z^2}},$$

$$\therefore \frac{dx}{x\sqrt{1-x^2}} = -\frac{dz}{1-z^2},$$

---

* If $m = 0$, the formula fails; for the second member becomes infinite; in this case, therefore, we must have recourse to one of the elementary integrals.

which being integrated (Sect. III. Ex. 1.), gives

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\frac{1}{2} l \frac{1+z}{1-z},$$

$$= -\frac{1}{2} l . \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}},$$

which, by multiplying its terms by the numerator, gives, after reduction,

$$\int \frac{dx}{x\sqrt{1-x^2}} = -l \frac{1+\sqrt{1-x^2}}{x}.$$

Hence by successively substituting for m, in the proposed differential, the numbers of the series 1, 3, 5, ... we find

$$\int \frac{dx}{x\sqrt{1-x^2}} = -l \frac{1+\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^3\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{2x^2} - \frac{1}{2} l \frac{1+\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^5\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{4x^4} - \frac{3}{4}\frac{\sqrt{1-x^2}}{2x^2} - \frac{1.3}{2.4} l \frac{1+\sqrt{1-x^2}}{x},$$

. . . .

And by the successive substitution of the numbers of the series 2, 4, ... we find

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^4\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{3x^3} - \frac{2}{3}\frac{\sqrt{1-x^2}}{x},$$

$$\int \frac{dx}{x^6\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{5x^5} - \frac{4}{5}\frac{\sqrt{1-x^2}}{3x^3} - \frac{2.4}{3.5} \cdot \frac{\sqrt{1-x^2}}{x}.$$

. . . .

The following examples are added for the exercise of the student in the integration of irrational functions.

Ex. 11. Let $u\,dx = \dfrac{x^m dx}{(a+bx)^{\frac{3}{2}}}$,

$$a + bx = \mathrm{x},$$

$$\int \frac{dx}{\mathrm{x}^{\frac{3}{2}}} = -\frac{2}{b\sqrt{\mathrm{x}}},$$

$$\int \frac{x\,dx}{\mathrm{x}^{\frac{3}{2}}} = (\mathrm{x}+a)\frac{2}{b^2\sqrt{\mathrm{x}}},$$

$$\int \frac{x^2 dx}{\mathrm{x}^{\frac{3}{2}}} = (\tfrac{1}{3}\mathrm{x}^2 - 2a\mathrm{x} - a^2)\frac{2}{b^3\sqrt{\mathrm{x}}},$$

$$\int \frac{x^3 dx}{\mathrm{x}^{\frac{3}{2}}} = (\tfrac{1}{5}\mathrm{x}^3 - a\mathrm{x}^2 + 3a^2\mathrm{x} + a^3)\frac{2}{b^4\sqrt{\mathrm{x}}},$$

$$\int \frac{x^4 dx}{\mathrm{x}^{\frac{3}{2}}} = (\tfrac{1}{7}\mathrm{x}^4 - \tfrac{4}{5}a\mathrm{x}^3 + 2a^2\mathrm{x}^2 - 4a^3\mathrm{x} - a^4)\frac{2}{b^5\sqrt{\mathrm{x}}},$$

$$\int \frac{x^5 dx}{\mathrm{x}^{\frac{3}{2}}} = (\tfrac{1}{9}\mathrm{x}^5 - \tfrac{5}{7}a\mathrm{x}^4 + 2a^2\mathrm{x}^3 - \tfrac{10}{3}a^3\mathrm{x}^2 + 5a^4\mathrm{x} + a^5)\frac{2}{b^6\sqrt{\mathrm{x}}}.$$

Ex. 12. $u\,dx = \dfrac{dx}{x^m(a+bx)^{\frac{3}{2}}}$,

$$a + bx = \mathrm{x},$$

$$\int \frac{dx}{x\mathrm{x}^{\frac{3}{2}}} = \frac{2}{a\sqrt{\mathrm{x}}} + \frac{1}{a}\int \frac{dx}{x\sqrt{\mathrm{x}}},$$

$$\int \frac{dx}{x^2\mathrm{x}^{\frac{3}{2}}} = \left(-\frac{1}{ax} - \frac{3b}{a^2}\right)\frac{1}{\sqrt{\mathrm{x}}} - \frac{3b}{2a^2}\int \frac{dx}{x\sqrt{\mathrm{x}}},$$

$$\int \frac{dx}{x^3\mathrm{x}^{\frac{3}{2}}} = \left(-\frac{1}{2ax^2} + \frac{5b}{4a^2x} + \frac{15b^2}{4a^3}\right)\frac{1}{\sqrt{\mathrm{x}}} + \frac{15b^2}{8a^3}\int \frac{dx}{x\sqrt{\mathrm{x}}},$$

$$\int \frac{dx}{x^4\mathrm{x}^{\frac{3}{2}}} = \left(-\frac{1}{3ax^3} + \frac{7b}{12a^2x^2} - \frac{35b^2}{24a^3x} - \frac{35b^3}{8a^4}\right)\frac{1}{\sqrt{\mathrm{x}}} - \frac{35b^3}{16a^4}\int \frac{dx}{x\sqrt{\mathrm{x}}}.$$

Ex. 13. $udx = \dfrac{x^m dx}{(a+bx)^{\frac{5}{2}}}$,

$$a + bx = \mathrm{x},$$

$$\int \frac{dx}{\mathrm{x}^{\frac{5}{2}}} = -\frac{2}{3b\mathrm{x}\sqrt{\mathrm{x}}},$$

$$\int \frac{xdx}{\mathrm{x}^{\frac{5}{2}}} = (-\mathrm{x} + \tfrac{1}{3}a)\frac{2}{b^2\mathrm{x}\sqrt{\mathrm{x}}},$$

$$\int \frac{x^2dx}{\mathrm{x}^{\frac{5}{2}}} = (\mathrm{x}^2 + 2a\mathrm{x} - \tfrac{1}{3}a^2)\frac{2}{b^3\mathrm{x}\sqrt{\mathrm{x}}},$$

$$\int \frac{x^3dx}{\mathrm{x}^{\frac{5}{2}}} = (\tfrac{1}{3}\mathrm{x}^3 - 3a\mathrm{x}^2 - 3a^2\mathrm{x} + \tfrac{1}{3}a^3)\frac{2}{b^4\mathrm{x}\sqrt{\mathrm{x}}},$$

Ex. 14. $udx = \dfrac{x^m dx}{(a+bx)^{\frac{7}{2}}}$,

$$a + bx = \mathrm{x},$$

$$\int \frac{dx}{\mathrm{x}^{\frac{7}{2}}} = -\frac{2}{5b\mathrm{x}^2\sqrt{\mathrm{x}}},$$

$$\int \frac{xdx}{\mathrm{x}^{\frac{7}{2}}} = (-\tfrac{1}{3}\mathrm{x} + \tfrac{1}{5}a)\frac{2}{b^2\mathrm{x}^2\sqrt{\mathrm{x}}},$$

$$\int \frac{x^2dx}{\mathrm{x}^{\frac{7}{2}}} = (-\mathrm{x}^2 + \tfrac{2}{3}a\mathrm{x} - \tfrac{1}{5}a^2)\frac{2}{b^3\mathrm{x}^2\sqrt{\mathrm{x}}},$$

$$\int \frac{x^3dx}{\mathrm{x}^{\frac{7}{2}}} = (\mathrm{x}^3 + 3a\mathrm{x}^2 - a^2\mathrm{x} + \tfrac{1}{5}a^3)\frac{2}{b^4\mathrm{x}^2\sqrt{\mathrm{x}}}.$$

Ex. 15. $udx = x^m dx\sqrt{(a + bx)}$,

$$a + bx = \mathrm{x},$$

$$\int dx\sqrt{\mathrm{x}} = \frac{2\mathrm{x}\sqrt{\mathrm{x}}}{3b},$$

$$\int xdx\sqrt{\mathrm{x}} = (\tfrac{1}{5}\mathrm{x} - \tfrac{1}{3}a)\frac{2\mathrm{x}\sqrt{\mathrm{x}}}{b^2},$$

$$\int x^2dx\sqrt{\mathrm{x}} = (\tfrac{1}{7}\mathrm{x}^2 - \tfrac{2}{5}a\mathrm{x} + \tfrac{1}{3}a^2)\frac{2\mathrm{x}\sqrt{\mathrm{x}}}{b^3},$$

$$\int x^3 dx\sqrt{\mathrm{x}} = (\tfrac{1}{9}\mathrm{x}^3 - \tfrac{3}{7}a\mathrm{x}^2 + \tfrac{3}{5}a^2\mathrm{x} - \tfrac{1}{3}a^5)\frac{2\mathrm{x}\sqrt{\mathrm{x}}}{b^4}.$$

Ex. 16. $udx = \dfrac{dx\sqrt{(a+bx)}}{x^m}$,

$$a + bx = \mathrm{x},$$

$$\int\frac{dx\sqrt{\mathrm{x}}}{x} = 2\sqrt{\mathrm{x}} + a\int\frac{dx}{x\sqrt{\mathrm{x}}},$$

$$\int\frac{dx\sqrt{x}}{x^2} = -\frac{\sqrt{\mathrm{x}}}{x} + \frac{b}{2}\int\frac{dx}{x\sqrt{\mathrm{x}}},$$

$$\int\frac{dx\sqrt{\mathrm{x}}}{x^3} = -\frac{\mathrm{x}\sqrt{\mathrm{x}}}{2ax^2} + \frac{b\sqrt{\mathrm{x}}}{4ax} - \frac{b^2}{8a}\int\frac{dx}{x\sqrt{\mathrm{x}}},$$

$$\int\frac{dx\sqrt{\mathrm{x}}}{x^4} = \left(-\frac{1}{3ax^3} + \frac{b}{4a^2x^2}\right)\mathrm{x}\sqrt{\mathrm{x}} - \frac{b^2\sqrt{\mathrm{x}}}{8a^2x} + \frac{b^3}{16a^2}\int\frac{dx}{x\sqrt{\mathrm{x}}}.$$

Ex. 17. $udx = \dfrac{dx(a+bx)^{\frac{3}{2}}}{x^m}$,

$$a + bx = \mathrm{x},$$

$$\int\frac{dx\mathrm{x}^{\frac{3}{2}}}{x} = (\tfrac{1}{3}\mathrm{x} + a)2\sqrt{\mathrm{x}} + a^2\int\frac{dx}{x\sqrt{\mathrm{x}}},$$

$$\int\frac{dx\mathrm{x}^{\frac{3}{2}}}{x^2} = -\frac{\mathrm{x}^2\sqrt{\mathrm{x}}}{ax} + \frac{3b}{3a}\int\frac{dx\mathrm{x}^{\frac{3}{2}}}{x},$$

$$\int\frac{dx\mathrm{x}^{\frac{3}{2}}}{x^3} = \left(-\frac{1}{2ax^2} - \frac{b}{4a^2x}\right)\mathrm{x}^2\sqrt{\mathrm{x}} + \frac{3b^2}{8a^2}\int\frac{dx\mathrm{x}^{\frac{3}{2}}}{x},$$

$$\int\frac{dx\mathrm{x}^{\frac{3}{2}}}{x^4} = \left(-\frac{1}{3ax^3} + \frac{b}{12a^2x^2} + \frac{b^2}{24a^3x}\right)\mathrm{x}^2\sqrt{\mathrm{x}} - \frac{b^3}{16a^3}\int\frac{dx\mathrm{x}^{\frac{3}{2}}}{x}.$$

Ex. 18. $udx = \dfrac{dx}{(a+bx^2)^{\frac{n}{2}}}$,

$$a + bx^2 = \mathrm{x},$$

$$\int\frac{dx}{\mathrm{x}^{\frac{1}{2}}} = \int\frac{dx}{\sqrt{\mathrm{x}}},$$

$$\int\frac{dx}{\mathrm{x}^{\frac{3}{2}}} = \frac{x}{a\sqrt{\mathrm{x}}},$$

$$\int \frac{dx}{x^{\frac{5}{2}}} = \left(\frac{1}{3ax} + \frac{2}{3a^2}\right)\frac{x}{\sqrt{x}},$$

$$\int \frac{dx}{x^{\frac{7}{2}}} = \left(\frac{1}{5ax^2} + \frac{4}{15a^2x} + \frac{1}{15a^3}\right)\frac{x}{\sqrt{x}},$$

In general,

$$\int \frac{dx}{\sqrt{(a+bx^2)}} = \frac{1}{\sqrt{b}}\log.\ [x\sqrt{b} + \sqrt{(a+bx^2)}],$$

$$\text{or} \int \frac{dx}{\sqrt{(a+bx^2)}} = \frac{1}{\sqrt{-b}}\ \sin.^{-1} x\sqrt{-\frac{b}{a}},$$

The first expression is real when b is positive; the second when b is negative. Both a and b cannot be negative at the same time. Hence, we have

$$\text{I.} \int \frac{dx}{\sqrt{(\pm a + bx^2)}} = \frac{1}{\sqrt{b}}\log.[x\sqrt{b} + \sqrt{(\pm a + bx^2)}],$$

$$\text{II.} \int \frac{dx}{\sqrt{(a-bx^2)}} = \frac{1}{\sqrt{b}}\ \sin.^{-1} x\sqrt{\frac{b}{a}} = \frac{1}{\sqrt{b}}\ \cos.^{-1}\sqrt{\frac{a-bx^2}{a}},$$

$$= \frac{1}{2\sqrt{b}}\ \cos.^{-1}\frac{a-2bx^2}{a} = \frac{1}{\sqrt{b}}\ \text{tang.}^{-1}\frac{x\sqrt{b}}{\sqrt{(a-bx)}},$$

$$= \frac{1}{\sqrt{b}}\ \cot.^{-1}\frac{\sqrt{(a-bx^2)}}{x\sqrt{b}} = \frac{1}{\sqrt{b}}\ \sec.^{-1}\sqrt{\frac{a}{a-bx^2}},$$

$$= \frac{1}{\sqrt{b}}\ \text{cosec.}^{-1}\sqrt{\frac{a}{bx^2}} = \frac{1}{2\sqrt{b}}\ \text{vers. sin.}^{-1}\frac{2bx^2}{a}.$$

All these circular arcs vanish when $x = 0$.

Particular cases are

$$\int \frac{dx}{\sqrt{(1+x^2)}} = \log.\ [x + \sqrt{(1+x^2)}],$$

$$\int \frac{dx}{\sqrt{(x^2-1)}} = \log.\ [x + \sqrt{(x^2-1)}],$$

$$\int \frac{dx}{\sqrt{(1-x^2)}} = \sin.^{-1}x = \cos.^{-1}\sqrt{(1-x^2)} = \tfrac{1}{2}\cos^{-1}(1-2x^2),$$

$$= \text{tang.}^{-1}\frac{x}{\sqrt{(1-x^2)}} = \cot.^{-1}\frac{\sqrt{(1-x^2)}}{x} = \sec.^{-1}\frac{1}{\sqrt{(1-x^2)}}$$

$$= \text{cosec.}^{-1}\frac{1}{x} = \tfrac{1}{2}\text{vers. sin.}^{-1}2x^2.$$

The integral $\int \frac{dx}{\sqrt{(\pm a + bx^2)}}$ can only vanish on the supposition that $x = 0$, when the upper sign is taken, and in this case

$$\int \frac{dx}{\sqrt{(+a+bx^2)}} = \frac{1}{\sqrt{b}} \log. \left(x\sqrt{\frac{b}{a}} + \sqrt{\frac{a+bx^2}{a}} \right).$$

Ex. 19. $udx = \dfrac{x^m dx}{\sqrt{(a+bx^2)}}$,

$$a + bx^2 = \mathrm{X},$$

$$\int \frac{dx}{\sqrt{\mathrm{X}}} = \int \frac{dx}{\sqrt{\mathrm{X}}},$$

$$\int \frac{xdx}{\sqrt{\mathrm{X}}} = \frac{\sqrt{\mathrm{X}}}{b},$$

$$\int \frac{x^2dx}{\sqrt{\mathrm{X}}} = \frac{x\sqrt{\mathrm{X}}}{2b} - \frac{a}{2b}\int \frac{dx}{\sqrt{\mathrm{X}}},$$

$$\int \frac{x^3dx}{\sqrt{\mathrm{X}}} = \left(\frac{x^2}{3b} - \frac{2a}{3b^2}\right)\sqrt{\mathrm{X}},$$

$$\int \frac{x^4dx}{\sqrt{\mathrm{X}}} = \left(\frac{x^3}{4b} - \frac{3ax}{8b^2}\right)\sqrt{\mathrm{X}} + \frac{3a^2}{8b^2}\int \frac{dx}{\sqrt{\mathrm{X}}},$$

$$\int \frac{x^5dx}{\sqrt{\mathrm{X}}} = \left(\frac{x^4}{5b} - \frac{4ax^2}{15b^2} + \frac{8a^2}{15b^3}\right)\sqrt{\mathrm{X}}.$$

Ex. 20. $udx = \dfrac{dx}{x^m\sqrt{(a+bx^2)}}$,

$$a + bx^2 = \mathrm{X},$$

$$\int \frac{dx}{x\sqrt{\mathrm{X}}} = \int \frac{dx}{x\sqrt{\mathrm{X}}},$$

$$\int \frac{dx}{x^2\sqrt{\mathrm{X}}} = -\frac{\sqrt{\mathrm{X}}}{ax},$$

$$\int \frac{dx}{x^3\sqrt{\mathrm{X}}} = -\frac{\sqrt{\mathrm{X}}}{2ax^2} - \frac{b}{2a}\int \frac{dx}{x\sqrt{\mathrm{X}}},$$

$$\int \frac{dx}{x^4\sqrt{\mathrm{X}}} = \left(-\frac{1}{3ax^3} + \frac{2b}{3a^2x}\right)\sqrt{\mathrm{X}}.$$

In general,

$$\int \frac{dx}{x\sqrt{(a+bx^2)}} = \frac{1}{2\sqrt{a}} \log. \frac{\sqrt{(a+bx^2)} - \sqrt{a}}{\sqrt{(a+bx^2)} + \sqrt{a}},$$

$$\text{or } \int \frac{dx}{x\sqrt{(a+bx^2)}} = \frac{1}{\sqrt{-a}} \sec.^{-1} x\sqrt{\left(-\frac{b}{a}\right)},$$

the first of which is real when a is positive; the second when a is negative: a and b cannot both be negative at the same time.

$$\text{I. } \int \frac{dx}{x\sqrt{(+a+bx^2)}} = \frac{1}{2\sqrt{a}} \log. \frac{\sqrt{(a+bx^2)} - \sqrt{a}}{\sqrt{(a+bx^2)} + \sqrt{a}},$$

$$= \frac{1}{\sqrt{a}} \log. \frac{\sqrt{(a+bx^2)} - \sqrt{a}}{x},$$

where $\sqrt{a}$ may be positive or negative. This integral cannot vanish when $x = 0$.

$$\text{II. } \int \frac{dx}{x\sqrt{(-a+bx^2)}} = \frac{1}{\sqrt{a}} \sec.^{-1} x \sqrt{\frac{b}{a}} =$$

$$\frac{1}{\sqrt{a}} \tang.^{-1} \sqrt{\frac{bx^2 - a}{a}}.$$

$$= \frac{1}{\sqrt{a}} \cot.^{-1} \sqrt{\frac{a}{bx^2 - a}} = \frac{1}{\sqrt{a}} \operatorname{cosec.}^{-1} \frac{x\sqrt{b}}{\sqrt{(bx^2 - a)}},$$

$$= \frac{1}{\sqrt{a}} \cos.^{-1} \frac{\sqrt{a}}{x\sqrt{b}} = \frac{1}{2\sqrt{a}} \cos.^{-1} \frac{2a - bx^2}{bx^2},$$

$$= \frac{1}{\sqrt{a}} \sin.^{-1} \frac{\sqrt{(bx^2 - a)}}{x\sqrt{b}} = \frac{1}{2\sqrt{a}} \text{vers. sin.}^{-1} \frac{2(bx^2 - a)}{bx^2}$$

All these integrals vanish, when $x = \sqrt{\frac{a}{b}}$; when $x = 0$ they cannot vanish.

Particular cases are

$$\int \frac{dx}{x\sqrt{(1+x^2)}} = \log. \frac{\sqrt{(1+x^2)} - 1}{x},$$

$$\int \frac{dx}{x\sqrt{(1-x^2)}} = \log. \frac{\sqrt{(1-x^2)} - 1}{x} = \log. \frac{1 - \sqrt{(1-x^2)}}{x},$$

$$\int \frac{dx}{x\sqrt{(x^2-1)}} = \sec.^{-1} x = \tan.^{-1}\sqrt{(x^2-1)} = \cot.^{-1}\sqrt{\frac{1}{x^2-1}},$$

$$= \operatorname{cosec.}^{-1}\frac{x}{\sqrt{(x^2-1)}} = \cos.^{-1}\frac{1}{x} = \tfrac{1}{2}\cos.^{-1}\frac{2-x^2}{x^2},$$

$$= \sin.^{-}\frac{\sqrt{(x^2-1)}}{x} = \tfrac{1}{2}\text{ver. }\sin.^{-1}\frac{2(x^2-1)}{x^2}.$$

SECTION VI.

Integration by series.

(230.) A series representing the integral of any differential may always be found by developing the differential coefficient in a series of powers of the variable, and integrating each term of the series after multiplying by dx. Thus, if X represent any function of x, and

$$\mathrm{X} = \mathrm{A}x^a + \mathrm{B}x^b + \mathrm{C}x^c \cdot \cdot \cdot \cdot$$

$$\int \mathrm{X}dx = \frac{\mathrm{A}x^{a+1}}{a+1} + \frac{\mathrm{B}x^{b+1}}{b+1} + \frac{\mathrm{C}x^{c+1}}{c+1} \cdot \cdot \cdot \cdot$$

Although such a series is always an analytical representation of the integral, yet it is of no use in obtaining a value, or approximate value of it, except when it converges. If the value assigned to the variable be very small, this will be the case if the exponents continually increase, or if the series ascends. But if the value assigned to the variable be very great, it will only be the case when the series descends, and involves negative powers of the variable.

Various analytical contrivances have therefore been used for developing functions in series of these kinds.

This method of integration is also useful even where the

can be assigned by other methods under a finite

two integrals thus found be equated, a develop- the finite integral will be obtained, and, in general, cess is attended with much greater facility than the evelopment of the integral itself. We shall there- e examples of this method of development and in- ı.

PROP. LXXX.

To develop an arc ϕ in a series of powers of its

n. $\phi = x$, $\because d\phi = \dfrac{dx}{\sqrt{1-x^2}}$. Let $(1 - x^2)^{-\frac{1}{2}}$ be

d by the binomial theorem,

$$^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \frac{1.3.5.7}{2.4.6.8}x^8 \ldots.$$

ying by dx, and integrating both sides,

$$= \frac{x}{1} + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} \ldots.$$

the development required. No constant is added, $= 0$ renders $\phi = 0$.

PROP. LXXXI.

To develop an arc in a series of powers of its

$= \tan.^{-1}x$, $\because d\phi = \dfrac{dx}{1+x^2}$. Developing $\dfrac{1}{1+x^2}$ by division, we find

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \ldots .$$

Multiplying by dx and integrating, we find

$$\varphi = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots .$$

No constant is added, since when $\varphi = 0$, $x = 0$.

PROP. LXXXII.

(233.) *To develop an arc in a series of powers of its cosine or cotangent.*

If $\varphi = \cos.^{-1}x$, $\therefore d\varphi = -\frac{dx}{\sqrt{1-x^2}}$. And if $\varphi' = \cot.^{-1}x$, $\therefore d\varphi' = -\frac{dx}{1+x^2}$. Hence the developments in these cases differ only in sign from the two former. But since in these cases φ and x do not vanish together, it is necessary that a constant should be introduced. Let this be c, $\therefore$ in the first case

$$\varphi = \text{c} - \sin.^{-1}x,$$

$$\therefore \cos.^{-1}x + \sin.^{-1}x = \text{c}, \therefore \text{c} = \frac{\pi}{2}.$$

In the second case, also

$$\cot.^{-1}x = \text{c} - \tan.^{-1}x,$$

$$\therefore \text{c} = \frac{\pi}{2}.$$

The sought developments are therefore

$$\varphi = \frac{\pi}{2} - \frac{x}{1} - \frac{x^3}{1.2.3} - \frac{1.3}{2.4}.\frac{x^5}{5} \ldots .$$

$$\varphi' = \frac{\pi}{2} - \frac{x}{1} + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} \ldots .$$

PROP. LXXXIII.

(234.) *To develop the versed sine of an arc in a series of powers of the arc itself.*

Let $\varphi = \text{ver. sin.}^{-1}x$, $\because\ d\varphi = \dfrac{dx}{\sqrt{2x - x^2}} = \dfrac{dx}{\sqrt{2x}\cdot\sqrt{1-\frac{1}{2}x}}$,

Developing $(1 - \frac{1}{2}x)^{-\frac{1}{2}}$ by the binomial theorem, we find

$$\frac{1}{\sqrt{1-\frac{1}{2}x}} = 1 + \frac{1}{2}\cdot\frac{x}{2} + \frac{1.3}{2.4}\cdot\frac{x^2}{4} + \frac{1.3.5}{2.4.6}\cdot\frac{x^3}{8}\ \cdots\cdot$$

Multiplying both sides by $\dfrac{dx}{\sqrt{2x}}$, and integrating, we find

$$\varphi = \frac{\sqrt{x}}{\sqrt{2}}\left\{2 + \frac{1}{2}\cdot\frac{x}{3} + \frac{1.3}{2.4}\cdot\frac{x^2}{5.2} + \frac{1.3.5}{2.4.6}\cdot\frac{x^2}{7.4}\ \cdots\cdot\right\}.$$

No constant is added, because when $x = 0$, $\varphi = 0$.

PROP. LXXXIV.

(235.) *To develop the logarithm of a given number in a series.*

Let the given number be x, we have $dl(1+x) = \dfrac{dx}{1+x}$. By division

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4\ \cdots\cdot$$

$$\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4}\ \cdots\cdot$$

Multiplying each of these by dx, and integrating, we find

$$l(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots$$

$$l(x+1) = lx + \frac{x^{-1}}{1} - \frac{x^{-2}}{2} + \frac{x^{-3}}{3} - \frac{x^{-4}}{4} \dots$$

Hence by subtraction

$$lx = \frac{x^1 - x^{-1}}{1} - \frac{x^2 - x^{-2}}{2} + \frac{x^3 - x^{-3}}{3} - \frac{x^4 - x^{-4}}{4}.$$

(236.) We shall now give some examples of integration by converging series.

Ex. 1. To integrate $\frac{dx}{\sqrt{x^2-1}}$, x being a high number.

Developing by the binomial theorem, and integrating, we find

$$\int \frac{dx}{\sqrt{x^2-1}} = lx - \frac{1}{1.2.x^2} - \frac{1.3}{2.4}.\frac{1}{4x^4} - \frac{1.3.5}{2.4.6}.\frac{1}{6x^6} \dots$$

This series converges rapidly.

Ex. 2. To integrate the same formula when x is nearly equal to unity. Let $x = 1 + u$, $\because$

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \frac{du}{\sqrt{2u+u^2}} = \frac{1}{\sqrt{2}} \int u^{-\frac{1}{2}} du \left(1 + \frac{u}{2}\right)^{-\frac{1}{2}},$$

developing $\left(1 + \frac{u}{2}\right)^{-\frac{1}{2}}$, and multiplying each term of the development by $u^{-\frac{1}{2}}du$, and integrating, we find

$$\int \frac{du}{\sqrt{2u+u^2}} = \sqrt{2u}\left(1 - \frac{1}{2.3}.\frac{u}{2} + \frac{1.3}{2.4.5}.\frac{u^2}{4} - \frac{1.3.5}{2.4.6.7}.\frac{u^3}{8} \dots\right)$$

Since $u = x - 1$, this series converges very rapidly.

Ex. 3. To integrate $\frac{\sqrt{1-e^2x^2}}{\sqrt{1-x^2}}dx$ by a series, e being very small. By the binomial theorem

$$\sqrt{1-e^2x^2} = 1 - \tfrac{1}{2}e^2x^2 - \frac{1.1}{2.4}e^4x^4 - \frac{1.1.3}{2.4.6}e^6x^6 - \dots$$

the series to be integrated is therefore

$$\int \frac{dx}{\sqrt{1-x^2}} \left\{ 1 - \frac{1}{2} e^2x^2 - \frac{1.1}{2.4} . e^4x^4 - \frac{1.1.3}{2.4.6} e^6x^6 - \dots \right\}$$

Each term of this development comes under the form

$$\int \frac{x^m dx}{\sqrt{1-x^2}},$$

which has been integrated in Sect. V. Ex. 9. Substituting therefore for

$$\int \frac{dx}{\sqrt{1-x^2}}, \quad \int \frac{x^2dx}{\sqrt{1-x^2}}, \quad \int \frac{x^4dx}{\sqrt{1-x^2}} \dots$$

their values thus determined, we find

$$\int \frac{dx\sqrt{1-e^2x^2}}{\sqrt{1-x^2}} = \sin.^{-1}x + \frac{1}{2}e^2 \left[\frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\sin.^{-1}x\right.$$

$$+ \frac{1.1}{2.4} . e^4 \left[\left(\frac{1}{4}x^3 + \frac{1.3}{2.4}x\right)\sqrt{1-x^2} - \frac{1.3}{2.4}\sin.^{-1}x\right]$$

$$+ \frac{1.1.3}{2.4.6} e^6 \left[\left(\frac{1}{6}x^5 + \frac{1.5}{4.6}x^3 + \frac{1.3.5}{2.4.6}x\right)\sqrt{1-x^2} - \frac{1.3.5}{2.4.6}\sin.^{-1}x\right]$$

$$+ \dots \dots \dots$$

Ex. 4. To integrate the formula

$$\frac{dx}{\sqrt{(2cx-x^2)(b-x)}}.$$

Developing, we find

$$(b-x)^{-\frac{1}{2}} = b^{-\frac{1}{2}}\left(1 - \frac{x}{b}\right)^{-\frac{1}{2}}$$

$$= b^{-\frac{1}{2}} \left\{ 1 + \frac{1}{2} . \frac{x}{b} + \frac{1.3}{2.4} . \frac{x^2}{b^2} + \frac{1.3.5}{2.4.6} . \frac{x^3}{b^3} \dots \right\}$$

The question will then be resolved by the integration of a series of differentials of the form

$$\frac{x^m dx}{\sqrt{2cx - x^2}},$$

which is done by the second case of (220.).

(237.) The following is a very general method of approximating to the values of integrals by series.

Let $z = \int u dx + c$, c being the arbitrary constant; and let z' be what z becomes when x becomes $x + h$. Now, since

$$\frac{dz}{dx} = u, \therefore \frac{d^2z}{dx^2} = \frac{du}{dx},$$

and, in general, $\frac{d^nz}{dx^n} = \frac{d^{n-1}u}{dx^{n-1}}$. Hence by Taylor's series

$$z' = z + u \cdot \frac{h}{1} + \frac{du}{dx} \cdot \frac{h^2}{1.2} + \frac{d^2u}{dx^2} \cdot \frac{h^3}{1.2.3} \cdots$$

$$\therefore z' - z = u \cdot \frac{h}{1} + \frac{du}{dx} \cdot \frac{h^2}{1.2} + \frac{d^2u}{dx^2} \cdot \frac{h^3}{1.2.3} \cdots$$

The arbitrary constant disappears in this series, because it is united to both z' and z by the same sign.

This series only expresses the difference between the values of the sought integral, corresponding to the values $x + h$ and x of the variable. Therefore, the integral itself is so far indeterminate. But it may be observed, that, by whatever process the integral may be found, it is in this respect equally indeterminate; for the arbitrary constant being necessary to complete its value, all that is in any case obtained is the difference between the whole integral and its value when it becomes equal to the arbitrary constant. In the present instance, the integral is said to be obtained *between the limits* x and $x + h$. For when $h = 0$ the series vanishes, and it gradually increases with h until x becomes $x + h$, h assuming some proposed value.

(238.) The value of the variable x, which makes the integral vanish, is said to be the *origin* of the integral. When the *limits* of an integral are not assigned or known, it is called an *indefinite integral*. Thus all integrals in which the value of the constant is not known, are *indefinite integrals*. But when the limits are assigned, they are then called complete or *definite integrals*.

(239.) In the preceding case, if the limits of the integral

be supposed to be $x = a$ and $x = b$, the value expressed in a series will be

$$\text{A} \cdot \frac{(b-a)}{1} + \text{A}' \cdot \frac{(b-a)^2}{1.2} + \text{A}'' \cdot \frac{(b-a)^3}{1.2.3} \dots$$

where A, A$'$, A$''$ .... are what u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$ ....

become when $x = a$.

(240.) The series

$$z' - z = u \cdot \frac{h}{1} + \frac{du}{dx} \cdot \frac{h^2}{1.2} + \frac{d^2u}{dx^2} \cdot \frac{h^3}{1.2.3} \dots$$

is only convergent when h is very small, and therefore would only determine the approximate value of the integral between very narrow limits. This inconvenience, however, is remedied by the successive application of it. Let z, A, A$'$, A$''$, ... be the values of z, u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, corresponding to x; z_1, A$_1$, A$_1'$, A$_1''$, ... those corresponding to $x + h$; z_2, A$_2$, A$_2'$, A$_2''$, ... those corresponding to $x + 2h$; and, in general, z_n, A$_n$, A$_n'$, A$_n''$, ... those corresponding to $x + nh$.

Hence we obtain the following series:

$$z_1 - z = \text{A} \cdot \frac{h}{1} + \text{A}' \cdot \frac{h^2}{1.2} + \text{A}'' \frac{h^3}{1.2.3} \dots$$

$$z_2 - z_1 = \text{A}_1 \cdot \frac{h}{1} + \text{A}_1' \frac{h^2}{1.2} + \text{A}_1'' \frac{h^3}{1.2.3} \dots$$

$$z_3 - z_2 = \text{A}_2 \cdot \frac{h}{1} + \text{A}_2' \cdot \frac{h^2}{1.2} + \text{A}_2'' \cdot \frac{h^3}{1.2.3} \dots$$

. . . .

. . . .

$$z_n - z_{n-1} = \text{A}_{n-1} \cdot \frac{h}{1} + \text{A}'_{n-1} \cdot \frac{h^2}{1.2.3} + \text{A}''_{n-1} \cdot \frac{h^3}{1.2.3} \dots$$

Let S(A) signify A + A$_1$ + A$_2$ .... A$_n$; and let a similar symbol express the sums of the other coefficients. By addition we find

$$z_n - z = \text{s}(\text{A}) \cdot \frac{h}{1} + \text{s}(\text{A}')\frac{h^2}{1.2} + \text{s}(\text{A}'')\cdot\frac{h^3}{1.2.3}\cdots$$

This series converges the more rapidly the smaller h is assumed. By these means we are enabled to integrate by series between any proposed limits. As before, let the limits be $x = a$ and $x = b$. Divide $b - a$ by n, and let $\frac{b-a}{n} = h$. The value of the integral will be

$$z_n - z = \frac{\text{s}(\text{A})}{1}\cdot\frac{b-a}{n} + \frac{\text{s}(\text{A}')}{1.2}\cdot\frac{(b-a)^2}{n^2} + \frac{\text{s}(\text{A}'')}{1.2.3}\cdot\frac{(b-a)^3}{n^3}\cdots\cdot$$

which may be made to converge with sufficient rapidity by assuming n sufficiently great.

It is obvious that this method becomes inapplicable if any exception of Taylor's series lie between the limits $x = a$ and $x = b$; which is indicated by some of the coefficients becoming infinite.

SECTION VII.

Of the integration of differentials whose coefficients are exponential or logarithmic functions of the variable.

(241.) The integration of transcendental functions is effected by the aid of the several formulæ already established for algebraic functions, united with some primitive formulæ peculiar to themselves, and derived immediately by inverting the rules for differentiation. These functions may assume such an infinite variety of forms, that no general methods of integration can be given; and, indeed, even were a classification possible, there are many formulæ whose integrals have not hitherto been assigned under a finite form. All, therefore, which an elementary work, such as the

present can effect, is to lay down the elementary integrals peculiar to each kind of transcendental functions, and explain the methods of integrating some of the most general formulæ by the union of these principles with those of algebraic functions. Every thing after this must depend upon the sagacity and expertness of the student in discovering transformations and artifices calculated to facilitate the integration of such formulæ as may occur, either by reducing them to others of a more simple form or more easily integrated, as we have done in Sect. IV., or by bringing them at once to an algebraic form. An approximation may *always* be had by the method of series explained in Sect. VI.

(242.) If $u = a^x$, $\therefore du = a^x l a dx$. Hence

$$\int a^x dx = \frac{a^x}{la},$$

this is the elementary integral of exponential functions.

(243.) A differential of the form $za^x dx$, where z is an algebraic function of a^x, may be integrated by making $a^x = u$, $\therefore$ the differential becomes $zudx$, but $udx = \frac{du}{la}$, $\therefore \int za^x dx = \frac{1}{la}\int \text{F}(u)du$, which may be integrated by the rules already given.

(244.) By differentiating ze^x, we find

$$d(ze^x) = ze^x dx + e^x dz;$$

and if $\frac{dz}{dx} = z'$, we have

$$d(ze^x) = e^x(z + z')dx.$$

So that every differential of the form $e^x\text{F}(x)dx$, and in which $\text{F}(x)$ consists of two parts, of which one (z') is the differential coefficient of the other (z), is easily integrated; for in that case, $\int e^x \text{F}(x)dx = e^x z$. An example will make this evident. Let the differential be

$$e^x(3x^2 + x^3 - 1)dx.$$

Now, since $\dfrac{d(x^3-1)}{dx} = 3x^2$, $\because$

$$\int e^x(3x^2 + x^3 - 1)dx = (x^3 - 1)e^x.$$

(245.) In most cases, however, it will be necessary to integrate by parts, and to establish formulæ of reduction by which the exponents of the functions which are involved in the differentials may be continually reduced.

(246.) To integrate the formula

$$udx = a^x x^n dx.$$

Integrating by parts, we find

$$\int udx = \frac{x^n a^x}{la} - \frac{n}{la}\int a^x x^{n-1} dx.$$

By successively substituting $n - 1$, $n - 2$, ... for n, the exponent of x will be reduced to 0, and the final integral will be $\int a^x dx = \dfrac{a^x}{la}$. This process gives

$$\int udx = \frac{a^x}{la}\left\{ x^n - \frac{n}{la}x^{n-1} + \frac{n \cdot n-1}{(la)^2}x^{n-2} - \frac{n \cdot n-1 \cdot n-2}{(la)^3}\cdot x^{n-3} \dots\dots \right\}$$

(247.) If the exponent n be negative, this series will not attain the desired end. In this case, however, by integration by parts, we find

$$\int \frac{a^x dx}{x^n} = -\frac{a^x}{(n-1)x^{n-1}} + \frac{la}{n-1}\int \frac{a^x dx}{x^{n-1}},$$

which produces a continual diminution in the exponent of x. The final integral in this case will be $\int \dfrac{a^x dx}{x}$. This integral has never been assigned under a finite form. It may, however, be developed in a series thus. By Maclaurin's theorem (63.),

$$\frac{a^x}{x} - \frac{1}{x} = \frac{la}{1} + \frac{(la)^2}{1.2}x + \frac{(la)^3}{1.2.3}\cdot x^2 + \dots.$$

Multiplying by dx, and integrating, we find

$$\int \frac{a^x dx}{x} = lx + \frac{la}{1} \cdot \frac{x}{1} + \frac{(la)^2}{1.2} \cdot \frac{x^2}{2} + \frac{(la)^3}{1.2.3} \cdot \frac{x^3}{3} + \cdots$$

If n be a fraction, it will be also necessary to complete the integration by a series.

All the preceding observations, *mutatis mutandis*, apply to the formula $\mathrm{x}a^x dx$, x being any algebraic function of x.

(248.) If $u = lx$, $\because du = \frac{dx}{x}$, $\therefore \int \frac{dx}{x} = lx$. This is the elementary integral of logarithmic functions.

(249.) Let $udx = \mathrm{x}(lx)^n dx$. By integration by parts, we find

$$\int udx = (lx)^n \int \mathrm{x} dx - n \int \frac{(lx)^{n-1} dx}{x} \int \mathrm{x} dx.$$

Since x is supposed to be an algebraic function of x, the integral $\int \mathrm{x} dx$ may be considered as known.

Thus, when n is a positive integer, the above formula will, by continual substitution for n, reduce the exponent of the logarithm.

But if n be negative, it will be necessary to integrate by parts in another way. Since

$$\int \frac{dx}{x} (lx)^n = \frac{(lx)^{n+1}}{n+1},$$

if the quantity $\mathrm{x}(lx)^n dx$ be supposed to consist of the factors $\mathrm{x}x$ and $\frac{(lx)^n}{x} dx$, we find

$$\int \frac{\mathrm{x}}{(lx)^n} dx = \frac{\mathrm{x}x}{1-n} (lx)^{1-n} - \frac{1}{1-n} \int (lx)^{1-n} d(\mathrm{x}x).$$

By the successive application of this formula, the integration will be reduced to that of the formula $lxd(\mathrm{x}x)$, which is of the form $\mathrm{x}' lx dx$, x' being an algebraic function of x. The integration of this will altogether depend upon the form of the function x'.

If n be an improper fraction, it may be reduced to a proper one; but the final integration must be effected by series.

SECTION VIII.

Praxis on the integration of exponential and logarithmic differentials.

Ex. 1. Let $udx = \frac{a^x dx}{\sqrt{1+a^{nx}}}$. Let $a^x = z$, $\therefore dx = \frac{dz}{zla}$, and $a^{nx} = z^n$. Hence the integration is reduced to

$$\int udx = \frac{1}{la}\int\frac{dz}{(1+z^n)^{\frac{1}{2}}},$$

which is integrated by series.

Ex. 2. Let $udx = \frac{e^x x dx}{(1+x)^2}$. Let $1+x=z$, $\therefore$

$$udx = \frac{e^z}{e}\left(\frac{dz}{z} - \frac{dz}{z^2}\right).$$

Since $d\frac{dz}{z} = -\frac{dz}{z^2}$, $\therefore$ by (244.),

$$\int udx = \frac{e^z}{ze} = \frac{e^x}{1+x}.$$

Ex. 3. Let $udx = x^m(lx)^n dx$. Since $\int x^m dx = \frac{x^{m+1}}{m+1}$, we find by (249.),

$$\int udx = \frac{x^{m+1}(lx)^n}{m+1} - \frac{n}{m+1}\int x^m(lx)^{n-1}\cdot dx.$$

Substituting for n successively $n-1$, $n-2$, &c. we find

$$\int u dx = \frac{x^{m+1}}{m+1} \cdot \left\{ (lx)^n - \frac{n}{m+1}(lx)^{n-1} + \frac{n \cdot (n-1)}{(m+1)^2}(lx)^{n-2} \right.$$
$$\left. - \frac{n \cdot n-1 \cdot n-2}{(m+1)^3}(lx)^{n-3} \dots \right\}.$$

This series is finite when n is a positive integer.

Thus, if $n = 1$, $= 2$, $= 3$, we find

$$\int x^m lx dx = \frac{x^{m+1}}{m+1}\left\{ lx - \frac{1}{m+1} \right\},$$

$$\int x^m (lx)^2 dx = \frac{x^{m+1}}{m+1}\left\{ (lx)^2 - \frac{2}{m+1}lx + \frac{1.2}{(m+1)^2} \right\},$$

$$\int x^m (lx)^3 dx = \frac{x^{m+1}}{m+1}\left\{ (lx)^3 - \frac{3}{m+1}(lx)^2 + \frac{2.3}{(m+1)^2}(lx) - \right.$$
$$\left. \frac{1.2.3}{(m+1)^3} \right\}.$$

This is subject to an exception when $m = -1$.

Ex. 4. Let $udx = \frac{(lx)^n}{x}dx$, n being a positive integer.

Since $\frac{dx}{x} = dlx$, $\because$

$$udx = (lx)^n d(lx),$$

$$\therefore \int udx = \frac{(lx)^{n+1}}{n+1}.$$

This is subject to an exception when $n = -1$. See Ex. 6.

Ex. 5. Let $udx = \frac{x^m dx}{lx}$. Let $z = x^{m+1}$, $\because x^m dx = \frac{dz}{m+1}$.

Also $lz = (m+1)lx$. Hence

$$udx = \frac{dz}{lz}.$$

Let $lz = y$, $\because z = e^y$ and $dz = e^y dy$. Hence

$$udx = \frac{e^y dy}{y}.$$

This is integrated by a series (247.).

Ex. 6. Let $udx = \frac{x^m dx}{(lx)^n}$, n being a positive integer. By (249.), we find

$$\int udx = -\frac{x^{m+1}}{(n-1)(lx)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m dx}{(lx)^{n-1}}.$$

By successively substituting $n-1$, $n-2$, &c. for n, we find

$$\int udx = x^{m+1}\left\{\frac{(lx)^{1-n}}{1-n} - \frac{(m+1)(lx)^{2-n}}{(1-n)(2-n)} + \frac{(m+1)^2(lx)^{3-n}}{(1-n)(2-n)(3-n)} \right.$$
$$\left. \cdots\cdots \right\} + \frac{(m+1)^{n-1}}{(n-1)(n-2)\cdots 1}\int \frac{x^m dx}{lx},$$

when $m = -1$, the formula of reduction becomes

$$\int \frac{dx}{x(lx)^n} = -\frac{1}{(n-1)(lx)^{n-1}} + \text{const.}$$

This is liable to an exception, since it becomes infinite when $n = 1$. The integral in this case is, however, easily obtained; for let

$$udx = \frac{dx}{xlx},$$

and let $z = lx$, $\therefore$

$$\int udx = \int \frac{dz}{z} = lz,$$

$$\therefore \int udx = l(lx) = l^2 x.$$

The final integral on which $\int \frac{x^m}{(lx)^n}dx$ depends appears, by the series just found, to be in general $\int \frac{x^m dx}{lx}$. Ex. 5.

Ex. 7. Let $udx = lxdx$. Integrating by parts

$$\int udx = xlx - x = x(lx - 1) = xl\left(\frac{x}{e}\right).$$

Ex. 8. Let $udx = \frac{dx}{xlx^2}$. Let $z = lx$, $\because \frac{dx}{x} = dz$, $\therefore$

$$udx = \frac{dz}{z^2},$$

$$\int udx = -\frac{1}{z} = -\frac{1}{lx}.$$

SECTION IX.

The integration of differentials whose coefficients are circular functions of the variable.

(250.) The elementary integrals on which the integration of circular functions depends, besides those of algebraic functions, are derived from the following differentials:

$$d \cdot \sin.n\phi = n\cos.n\phi d\phi, \quad d \,.\, \sec.n\phi = \frac{n\sin.n\phi}{\cos.^2 n\phi}d\phi,$$

$$d \cdot \cos.n\phi = - n\sin.n\phi \cdot d\phi, \quad d \cdot \cot.n\phi = -\frac{nd\phi}{\sin.^2 n\phi},$$

$$d \cdot \tan.n\phi = \frac{nd\phi}{\cos.^2 n\phi}, \quad d \operatorname{cosec}.n\phi = -\frac{n\cos.n\phi}{\sin.^2 n\phi}d\phi.$$

From which are derived the following formulæ:

$$\int \cos.n\phi d\phi = \frac{\sin.n\phi}{n}, \quad \int \frac{\sin.n\phi d\phi}{\cos.^2 n\phi} = \frac{\sec.n\phi}{n},$$

$$\int \sin.n\phi d\phi = -\frac{\cos.n\phi}{n}, \quad \int \frac{d\phi}{\sin.^2 n\phi} = -\frac{\cot.n\phi}{n},$$

$$\int \frac{d\phi}{\cos.^2 n\phi} = \frac{\tan.n\phi}{n}, \quad \int \frac{\cos.n\phi d\phi}{\sin.^2 n\phi} = -\frac{\operatorname{cosec}.n\phi}{n}.$$

(251.) When the arc or angle enters the differential coefficient, it is generally disengaged from it by integration by parts, either immediately or by the continual reduction of its exponent. The following formula will illustrate this:

$$\int \mathrm{x}\phi dx = \phi \int \mathrm{x}dx - \int d\phi \int \mathrm{x}dx.$$

Where x represents any algebraic function of x, and x re-

presents any trigonometrical function of the arc φ. Since $d\varphi$ must be an algebraic function of x, if φ be considered as a function of x, it follows that the integral $\int d\varphi \int \mathrm{x}dx$ comes under algebraic functions, and may accordingly be obtained by the rules already established. For examples of integrations thus effected, see the next section.

(252.) When the differential coefficient is a function of trigonometrical lines only, the integration may be effected by various analytical contrivances derived either from algebraic transformations or from trigonometrical formulæ. The following are the principal methods.

(253.) I. All functions whatever of trigonometrical lines may be reduced to functions of the sine and cosine. By these means the proposed formula may be transformed from a circular function to a differential of another kind. Let the proposed differential, when its coefficient has been reduced to a function of the sine or cosine, be $\mathrm{x}d\varphi$, x being a function of the sine or cosine. If $x = \sin.\varphi$, $\because\ d\varphi = \frac{dx}{\sqrt{1-x^2}}$, and if $x = \cos.\varphi$, $\because\ d\varphi = -\frac{dx}{\sqrt{1-x^2}}$; in either case the differential will, by the substitution thus suggested, be reduced to the form $\mathrm{x}dx$, x being a function of x, and may be integrated by the rules already established.

(254.) II. When powers of the sine or cosine of an arc occur in the differential coefficient, they may be developed in a series of the simple dimensions of the sines or cosines of multiples of that arc by the following well known trigonometrical series:

$$\sin.^n x = \mp \frac{1}{2^{n-1}} \left\{ \sin.nx - \frac{n}{1} \sin.(n-2)x + \frac{n.n-1}{1.2} \sin.(n-4)x \cdots \right\};$$

this applies to the case where n is odd, and the upper or lower sign is to be used according as $\frac{n-1}{2}$ is odd or even.

The series when n is even is

$$\sin.^n x = \mp \frac{1}{2^{n-1}}\left\{\cos.nx - \frac{n}{1}\cos.(n-2)x + \frac{n \cdot n-1}{1.2}\cos.(n-4)x \ldots\right\},$$

and $-$ or $+$ is used as $\frac{n}{2}$ is odd or even.

In general,

$$2^{n-1} \cdot \cos.^n x = \cos.nx + \frac{n}{1}\cos.(n-2)x + \frac{n \cdot n-1}{1.2}\cos.(n-4)x \ldots *.$$

(255.) III. When the sines and cosines are connected by multiplication in the differential coefficient, they may be disengaged by the formulæ,

$$2\sin.x\cos.y = \sin.(x+y) + \sin.(x-y),$$
$$2\sin.y\cos.x = \sin.(x+y) - \sin.(x-y),$$
$$2\cos.x\cos.y = \cos.(x+y) + \cos.(x-y),$$
$$2\sin.x\sin.y = \cos.(x-y) - \cos.(x+y).$$

(256.) IV. Functions of the sine or cosine may always be converted into exponential functions by the formulæ,

$$2\cos.x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}},$$
$$2\sqrt{-1}\sin.x = e^{x\sqrt{-1}} - e^{-x\sqrt{-1}},$$

which may be established thus; by (33.), Ex. 20, if

$$u = \cos.x \pm \sqrt{-1} \cdot \sin.x,$$

---

* This series terminates with the term in which the coefficient of x either vanishes or becomes $= 1$. In the former case, this term should be divided by two. See Trigonometry.

$$du = \pm \sqrt{-1} \cdot udx,$$

$$\therefore \frac{du}{u} = \pm\sqrt{-1} \cdot dx,$$

$$\therefore u = e^{\pm x\sqrt{-1}} + \text{const.}$$

But the constant must $= 0$, since, when $x = 0$, $u = 1$, $\therefore$

$$\cos.x \pm \sqrt{-1} \sin.x = e^{\pm x\sqrt{-1}}.$$

These two formulæ being added and subtracted, give the above mentioned results; and therefore their integration may be reduced to that of exponential functions.

(257.) V. When the differential is of the form

$$\sin.^m\varphi \cos.^n\varphi d\varphi,$$

it may be immediately reduced to a binomial differential, and integrated as in (221.) et seq. by putting

$$\sin.\varphi = x,$$

$$\cos.\varphi d\varphi = dx,$$

$$\therefore \sin.^m\varphi \cos.^n\varphi d\varphi = x^m(1 - x^2)^{\frac{n-1}{2}} dx.$$

Or by immediate integration by parts the three following pairs of formulæ may be readily obtained. In the first pair, one of the exponents is continually increased, while the other is diminished. In the second, one of the exponents is diminished, while the other is stationary; and in the third, one is increased, while the other is stationary.

$$\left.\begin{aligned}\int\sin^m\varphi \cos^n\varphi d\varphi &= \frac{\sin^{m+1}\varphi \cos^{n-1}\varphi}{m+1} + \frac{n-1}{m+1}\int\sin^{m+2}\varphi\cos^{n-2}\varphi d\varphi \\ \int\sin^m\varphi\cos^n\varphi d\varphi &= -\frac{\sin^{m-1}\varphi \cos^{n+1}\varphi}{n+1} + \frac{m-1}{n+1}\int\sin^{m-2}\varphi\cos^{n+2}\varphi d\varphi\end{aligned}\right\}$$

$$\left.\begin{aligned}\int\sin^m\varphi\cos^n\varphi d\varphi &= -\frac{\sin^{m-1}\varphi \cos^{n+1}\varphi}{m+n} + \frac{m-1}{m+n}\int\sin^{m-2}\varphi \cos^n\varphi d\varphi \\ \int\sin^m\varphi \cos^n\varphi d\varphi &= \frac{\sin^{m+1}\varphi \cos^{n-1}\varphi}{m+n} + \frac{n-1}{m+n}\int\sin^m\varphi \cos^{n-2}\varphi d\varphi\end{aligned}\right\}$$

$$\left.\begin{aligned}\int\sin^m\varphi\cos^n\varphi d\varphi &= \frac{\sin^{m+1}\varphi\cos^{n+1}\varphi}{m+1}+\frac{m+n+2}{m+1}\int\sin^{m+2}\varphi\cos^n\varphi d\varphi\\ \int\sin^m\varphi\cos^n\varphi d\varphi &= -\frac{\sin^{m+1}\varphi\cos^{n+1}\varphi}{n+1}+\frac{m+n+2}{n+1}\int\sin^m\varphi\cos^{n+2}\varphi d\varphi\end{aligned}\right\}$$

These formulæ are applicable, whatever be the values of the exponents m and n.

(258.) These methods united with integration by parts, will, in most cases, effect the integration of trigonometrical differentials. Much of the facility of the process must, however, depend on the expertness and ingenuity of the analyst, which is only to be acquired by practice, since no general methods can be assigned for obtaining the integrals of these functions. The processes for integrating several general and very useful formulæ are given in the following section.

SECTION X.

Praxis on the integration of circular functions.

Ex. 1. Let $udx = \frac{dx}{\sin.nx}$. Let $\cos.nx = z$,

$$\therefore dx = -\frac{1}{n}\frac{dz}{\sqrt{1-z^2}},$$

and $\sin.nx = \sqrt{1-z^2}$. Hence

$$\int\frac{dx}{\sin.nx} = -\frac{1}{n}\int\frac{dz}{1-z^2} = \frac{1}{n}\int\frac{dz}{z^2-1},$$

$$\therefore\int\frac{dx}{\sin.nx} = \frac{1}{n}\,l\,\frac{\sqrt{1-\cos.nx}}{\sqrt{1+\cos.nx}} = \frac{1}{n}l\tan.\tfrac{1}{2}nx.$$

Ex. 2. Let $udx = \frac{dx}{\cos.nx}$. In a similar way we find

$$\int \frac{dx}{\cos.nx} = \frac{1}{n} l \tan.\tfrac{1}{2}\left(\frac{\pi}{2} + nx\right) = l \tan.(45^0 + \tfrac{1}{2}nx).$$

Ex. 3. $udx = \dfrac{\cos.xdx}{\sin.x} = \cot.xdx.$ Since

$$d \sin.x = \cos.xdx, \because$$

$$\int \cot.xdx = \int \frac{d \sin.x}{\sin.x} = l \sin.x.$$

Ex. 4. $udx = \tan.xdx.$ Since $\sin.xdx = -d\cos.x,$

$$\therefore \int \tan.xdx = \int \frac{\sin.xdx}{\cos.x} = -\int \frac{d\cos.x}{\cos.x},$$

$$\therefore \int \tan.xdx = -l\cos.x = l \sec.x.$$

Ex. 5. $udx = \dfrac{dx}{\sin.x \cdot \cos.x}.$ Since $2 \sin.x \cos.x = \sin.2x, \because$

$$\int \frac{dx}{\sin.x \cdot \cos.x} = \int \frac{d(2x)}{\sin.2x} = l \tan.x \text{ (Ex. 1.)}.$$

Ex. 6. $udx = \dfrac{\sin.^{-1}x \cdot dx}{x}.$ Integrating by parts,

$$\int \frac{\sin.^{-1}xdx}{x} = \sin.^{-1}x \cdot lx - \int \frac{lxdx}{\sqrt{1-x^2}}.$$

Let $\varphi = \sin.^{-1}x, \because$

$$\int \frac{lx \cdot dx}{\sqrt{1-x^2}} = \int l \sin.\varphi d\varphi,$$

this may be integrated by a series.

Ex. 7. $udx = x^n \sin.^{-1}xdx.$ Integrating by parts,

$$\int x^n \sin.^{-1}xdx = \frac{x^{n+1}\sin.^{-1}x}{n+1} - \frac{1}{n+1}\int \frac{x^{n+1}dx}{\sqrt{1-x^2}}.$$

This integration fails when $n = -1.$ See Ex. 5.

Substituting successively $0, 1, 2, 3, \cdots$ for n, and replacing $\sin.^{-1}x$ and x by φ and $\sin.\varphi$, we find

$$\int \varphi d \sin.\varphi = \int \varphi \cos.\varphi d\varphi = \varphi \sin.\varphi + \cos.\varphi,$$
$$\int \varphi \sin.\varphi \cos.\varphi d\varphi = \tfrac{1}{2}\varphi \sin.^2\varphi + \tfrac{1}{4} \sin.\varphi \cos.\varphi - \tfrac{1}{4}\varphi,$$

$$\int\phi \sin.^2\phi \cos.\phi d\phi = \frac{1}{3}\phi \sin.^3\phi + \frac{1}{9}\sin.^2\phi \cos.\phi + \frac{1.2}{1.3.3}\cos.\phi,$$

$$\int\phi \sin.^3\phi \cos.\phi d\phi = \frac{1}{4}\phi \sin.^4\phi + \frac{1}{4\cdot 4}\sin.^3\phi \cos.\phi + \frac{1.3}{2.4.4}\cos.\phi - \frac{1.3}{2.4.4}\phi.$$

. . . .

Ex. 8. $udx = \sin.(m\phi + n)\cos.(p\phi + q)d\phi$. By (255.),

$$2\sin.(m\phi + n)\cos.(p\phi + q) = \sin.[(m + p)\phi + (n + q)] + \sin.[(m - p)\phi + (n - q)].$$

Multiplying by $d\phi$, and integrating by the formulæ (250.),

$$\int udx = -\frac{\cos.[(m+p)\phi + (n+q)]}{2(m+p)} - \frac{\cos.[(m-p)\phi + (n-q)]}{2(m-p)}.$$

Ex. 9. Let $udx = \sin.^n xdx$. Developing $\sin.^n x$ in a series, we find, when n is odd,

$$\sin.^n x = \mp\frac{1}{2^{n-1}}\left\{\sin.nx - \frac{n}{1}\sin.(n-2)x + \frac{n\cdot n-1}{1.2}\sin.(n-4)x \cdot\cdot\cdot\cdot\right\}.$$

Multiplying by dx, and integrating, we obtain

$$\int\sin.^n xdx = \pm\frac{1}{2^{n-1}}\left\{\frac{\cos.nx}{n} - \frac{n}{1}\cdot\frac{\cos.(n-2)x}{n-2} + \frac{n\cdot n-1}{1.2}\,\frac{\cos.(n-4)x}{n-4}\cdot\cdot\cdot\cdot\right\}.$$

In like manner, if n be even,

$$\int\sin.^n xdx = \mp\frac{1}{2^{n-1}}\left\{\frac{\sin.nx}{n} - \frac{n}{1}\,\frac{\sin.(n-2)x}{n-2} + \frac{n\cdot n-1}{1.2}\,\frac{\sin.(n-4)x}{n-4}\cdot\cdot\cdot\cdot\right\}.$$

By substituting successively for n the integers 1, 2, 3,.... we find

$$\int \sin.x dx = -\cos.x,$$
$$\int \sin.^2 x dx = -\tfrac{1}{4}\sin.2x + \tfrac{1}{2}x,$$
$$\int \sin.^3 x dx = \tfrac{1}{4}\cdot\frac{\cos.3x}{3} - \tfrac{3}{4}\cos.x,$$
$$\int \sin.^4 x dx = \tfrac{1}{8}\,\frac{\sin.4x}{4} - \tfrac{1}{4}\sin.2x + \tfrac{3}{8}x.$$

. . . .

Ex. 10. Let $udx = \cos.^n x \cdot dx$. Multiplying the series

$$2^{n-1}\cos.^n x = \cos.nx + \frac{n}{1}\cos.(n-2)x + \frac{n\cdot n-1}{1.2}\cos.(n-4)x +$$

by dx, and integrating, the result is

$$2^{n-1}\int \cos.^n x dx = \frac{\sin.nx}{n} + \frac{n}{1}\cdot\frac{\sin.(n-2)x}{n-2} +$$
$$\frac{n\cdot n-1}{1.2}\cdot\frac{\sin.(n-4)x}{n-4}\cdots$$

Hence, by substituting successively 1, 2, 3, ... for n, we find

$$\int \cos.x dx = \sin.x,$$
$$\int \cos.^2 x \cdot dx = \tfrac{1}{4}\sin.2x + \tfrac{1}{2}x,$$
$$\int \cos.^3 x dx = \tfrac{1}{12}\sin.3x + \tfrac{3}{4}\sin.x,$$
$$\int \cos.^4 x \cdot dx = \tfrac{1}{32}\sin.4x + \tfrac{1}{4}\sin.2x + \tfrac{3}{8}\cdot x.$$

. . . .

Ex. 11. Let $udx = \sin.x\cos.^n x dx$. Since $d\cos.x = -\sin.x dx$,

$$\int udx = -\int \cos.^n x\, d\cos.x = -\frac{\cos.^{n+1}x}{n+1}.$$

Ex. 12. Let $udx = \cos.x\sin.^n x \cdot dx$. This, in like manner, gives

$$\int udx = \frac{\sin.^{n+1}x}{n+1}.$$

Ex. 13. Let. $udx = \sin.^2 x\cos.^n x dx$. Let $\cos.x = z$, ∵

$$dz = -\sin.x dx,\ \sin.x = \sqrt{1-z^2},\ \therefore$$

$$\int u dx = -\int \frac{z^n dz}{\sqrt{1-z^2}},$$

which has already been integrated, Sect. V. Ex. 9.

SECTION XI.

Of successive integration.

(259.) When a differential coefficient is of an order superior to the first, as many successive integrations are necessary to arrive at the *integral*, or the primitive function from which it was derived, as there are units in the exponent of its order, and the same number of arbitrary constants will be introduced.

Let u be the integral, and let X be the differential coefficient of the nth order; so that

$$\frac{d^n u}{dx^n} = \mathrm{X}.$$

Let $d^{n-1}.u = p dx^{n-1}$. Hence

$$\frac{dp}{dx} = \mathrm{X},$$

$$\therefore dp = \mathrm{X} dx.$$

Let the integral of this, found by the methods already established, be

$$p = \mathrm{X}' + \mathrm{A},$$

A being an arbitrary constant. Hence

$$\frac{d^{n-1}u}{dx^{n-1}} = \mathrm{X}' + \mathrm{A}.$$

Again, let $d^{n-2}.u = p' dx^{n-2}$, $\therefore$

$$\frac{dp'}{dx} = \mathrm{X}' + \mathrm{A},$$

$$\therefore dp' = \text{x}'dx + \text{A}dx.$$

Integrating this as before, we find

$$p' = \text{x}'' + \text{A}x + \text{B},$$

$$\therefore \frac{d^{n-2}.u}{dx^{n-2}} = \text{x}'' + \text{A}x + \text{B},$$

where $\text{x}'' = \int \text{x}'dx$ and B is an arbitrary constant.

By applying to this a similar process, we find

$$\frac{d^{n-3}u}{dx^{n-3}} = \text{x}''' + \frac{\text{A}}{1.2}x^2 + \frac{\text{B}}{1}x + \text{C}.$$

And so on successively, we find

$$\frac{d^{n-4}u}{dx^4} = \text{x}'''' + \frac{\text{A}}{1.2.3}.x^3 + \frac{\text{B}}{1.2}.x^2 + \frac{\text{C}}{1}x + \text{D},$$

. . . .

. . . .

$$u = \text{x}^{(n)} + \frac{\text{A}}{1.2....n}x^{n-1} + \frac{\text{B}}{1.2....n-1}x^{n-2} + \dots$$

$$\dots \frac{\text{M}}{1.2}x^2 + \frac{\text{N}}{1}x + \text{P}.$$

It follows, therefore, that the integrals of all differentials of the same order agree in a number of terms expressed by the exponent of their order, and that the coefficients of these terms are arbitrary constants.

SECTION XII.

Of rectification, quadrature, and cubature.

I. *Rectification.*

(260.) If s express the arc of a plane curve related to rectangular co-ordinates, the differential of the arc is (126.),

$$ds = \sqrt{dy^2 + dx^2}.$$

By inverting this formula, we find an expression for the arc itself,

$$s = \int \sqrt{dy^2 + dx^2} = \int \sqrt{\frac{dy^2}{dx^2} + 1} \cdot dx.$$

By the equation of the curve, the value of $\frac{dy}{dx}$ may be obtained in terms of x, and the formula to be integrated in order to obtain the arc will assume the form $\text{x}dx$, x being a function of x; this being integrated between any proposed limits, will determine the corresponding arc of the curve.

The determination of the length of the arc of a curve is called *Rectification.* *Geometry* (329.).

(261.) If the curve be expressed by an equation related to polar co-ordinates, the radius vector being represented by r, and the variable angle by ω, we have, Geometry (329.),

$$s = \int \sqrt{r^2 d\omega^2 + dr^2},$$

$$\therefore s = \int \sqrt{r^2 + \frac{dr^2}{d\omega^2}} \cdot d\omega.$$

The coefficient $\frac{dr}{d\omega}$, and r being expressed as functions of ω, this formula may be integrated by the established rules; or if $\frac{dr}{d\omega}$ be expressed as a function of r, and $d\omega$ as a function of r and dr, the formula assumes the form $\text{R}dr$, R being a function of r; and, accordingly, we can integrate by the rules already known.

(262.) If a curve have double curvature, it must be expressed by two equations between three variables related to three axes of co-ordinates. In this case, the expression for the differential of the arc is

$$ds = \sqrt{dx^2 + dy^2 + dz^2},$$

$$\therefore ds = dz \sqrt{\frac{dx^2}{dz^2} + \frac{dy^2}{dz^2} + 1}.$$

Since by means of the two equations, x and y may be expressed as functions of z, the coefficients $\frac{dx}{dz}$, $\frac{dy}{dz}$, may also be obtained as functions of z, and therefore the formula to be integrated is reduced to the form $\mathrm{z}dz$, z being a function of z.

II. *Quadrature.*

(263.) If an area a be bounded by a plane curve related to rectangular co-ordinates, its differential is expressed thus, (127.),

$$da = ydx,$$
$$\therefore a = \int ydx.$$

The equation will determine y in terms of x, and the integral which determines the area assumes the form $\int \mathrm{x}dx$, which may be obtained within any proposed limits by the methods already established.

If the curve be related to polar co-ordinates, the area usually obtained is that included between two radii vectores. Its differential is expressed thus; Geometry (330.),

$$da = \tfrac{1}{2}r^2 d\omega,$$
$$\therefore a = \tfrac{1}{2}\int r^2 d\omega.$$

By the equation of the curve, r^2 may be obtained in terms of ω, and the integration may be effected by the established methods.

The determination of the area of any surface is called *Quadrature.*

(264.) If the surface of which the quadrature is sought be not plane, its equation must be expressed between three variables related to three axes of co-ordinates. In this case the differential of the area is

$$da = \sqrt{dx^2dy^2 + dy^2dz^2 + dx^2dz^2},$$

$$\therefore da = dxdy\sqrt{1 + \frac{dz^2}{dy^2} + \frac{dz^2}{dx^2}}.$$

The partial differential coefficients $\frac{dz}{dy}$ and $\frac{dz}{dx}$ may be obtained from the equation of the surface as functions of x and y; and thus the value of a is obtained by a double integration, first, with respect to x, and, secondly, with respect to y. For

$$a = \int dy \int dx \sqrt{1 + \frac{dz^2}{dy^2} + \frac{dz^2}{dx^2}}.$$

The integral $\int dx \sqrt{1 + \frac{dz^2}{dy^2} + \frac{dz^2}{dx^2}}$ may be found by considering x only as variable, and this being determined, the remaining integral is found by considering y alone variable.

This process may be illustrated by imagining the proposed surface divided into indefinitely minute rectangular spaces, any of which may be conceived to coincide with the tangent plane to the surface. Any one of these spaces may be taken as the differential of the area; and since it is equal to the square root of the sum of the squares of its projections on the three co-ordinate planes, the first of the preceding formulæ for da is immediately obtained. The first integration for x, considering y constant, gives the area of a zone of the surface intercepted by two planes perpendicular to the axis of y, and the perpendicular distance between which is dy. The final integration gives the sum of these zones, or the area of the surface intercepted between two planes perpendicular to the axis of y, and intersecting it at any two points whose distances from the origin are the limits of the integral.

(265.) A surface generated by the revolution of a plane curve round any line in its own plane as an axis, is called a

surface of revolution. The quadrature of such surfaces is effected with greater facility than other curved surfaces, since they require but one integration, and are expressed by the equations of their generatrices.

Let $y = \mathrm{F}(x)$ be the equation of the generatrix of a surface of revolution, the axis of revolution being assumed as axis of x, and the co-ordinates being rectangular. By the manner in which the surface is produced, it is evident that a section of it, by a plane perpendicular to the axis of x, and at any distance x from the origin, is a circle, the radius of which is y. The circumference of this circle is therefore $2\pi y$. If two such sections be imagined intercepting the arc ds of the generatrix, the area of the circular *zone* or *band* of the surface between them will obviously be $2\pi y ds$. This is therefore the differential of the surface; and if a be the area intercepted between two sections limited by any two values of x, ∵

$$a = 2\pi \int y ds,$$

the integral being taken between the proposed limits. By the equation of the generatrix ds may be obtained as a function of y and dy, or y and ds may be obtained as functions of x and dx. In either case the integration may be effected by the established methods.

III. *Cubature.*

(266.) The process by which the volume included by any given surface or surfaces is determined is called *Cubature.*

The equation $\mathrm{F}(xyz) = 0$ of a surface being given, we shall imagine it divided into laminæ by planes perpendicular to the axis of z. By assigning to z any given value z', and considering x and y to continue variable, the equation $\mathrm{F}(xyz') = 0$ will represent the plane curve produced by the section of the proposed surface by a plane through z' per-

pendicular to the axis of z. The quadrature of this section may be effected by the formula $\int y dx$, the value of y being obtained in terms of x and z' and the integration being made as if z' were constant. The integral in this case may be taken between any proposed limits. If this area be considered as the base of a lamina intercepted between two planes, the distance between which is dz, the volume of this lamina is $dz \int y dx$. This may be considered as the differential of the proposed volume, and the volume itself u will be

$$u = \int dz \int y dx = \int\int y dx dz.$$

(267.) If the solid be one of revolution round the axis of z, the area of the section perpendicular to the axis of z will be πy^2, $\therefore \int y dx = \pi y^2$. Hence the expression for the volume becomes in this case

$$u = \pi \int y^2 dz.$$

The equation of the generatrix between y and z being given, y^2 may be found in terms of z, and the integration effected between any proposed limits by the established methods.

SECTION XIII.

Examples of rectification, quadrature, and cubature.

PROP. LXXXV.

(268.) *To determine the arc of a parabolic curve represented by the equation* $y = px^n$.

By differentiating, we find

$$dy = npx^{n-1}dx,$$

$$\therefore dy^2 + dx^2 = (1 + n^2p^2x^{2(n-1)})dx^2,$$

$$\therefore s = \int (1 + n^2p^2x^{2(n-1)})^{\frac{1}{2}} \cdot dx^{\frac{1}{2}}.$$

This can be integrated in a finite form only when $2(n-1)$ is a submultiple of unity or of n (221). In other cases the integral may be expressed by a converging series.

If $n = \frac{3}{2}$, $\because 2n - 2 = 1$. In this case

$$s = \int (1 + \tfrac{9}{4}p^2 x)^{\frac{1}{2}} dx,$$

$$\because s = \frac{8}{27p^2}\left\{1 + \tfrac{9}{4}p^2 x\right\}^{\frac{3}{2}}.$$

The origin of this integral is $x = -\dfrac{4}{9p^2}$. This curve is the semicubical parabola, and is the evolute of the common parabola. (See Geometry, vol. i. (396.) and note.)

To determine the general class of parabolas which are rectifiable in finite terms, let m be an integer, and let $m = \dfrac{1}{2n-2}$, $\because n = \dfrac{1+2m}{2m}$. Hence $y = px^{\frac{1+2m}{2m}}$ represents the required class in this case. If $2(n-1)$ be a submultiple of n, let $m = \dfrac{n}{2(n-1)}$, m being an integer. Hence $n = \dfrac{2m}{2m-1}$, $\because y = px^{\frac{2m}{2m-1}}$. In general, therefore, the number n is a fraction, whose numerator exceeds its denominator by unity. If the denominator exceeds the numerator by unity, the integration may be effected by changing x into y, and *vice versa*.

(269.) If the curve be the common parabola $n = 2$, $\because$

$$ds = (1 + 4p^2x^2)^{\frac{1}{2}} dx.$$

Hence by the formula [2] (224.), we have

$$s = \tfrac{1}{2}x(1 + 4p^2x^2)^{\frac{1}{2}} + \tfrac{1}{2}\int \frac{dx}{\sqrt{1+4p^2x^2}};$$

but by Sect. V. Ex. 4 [2],

$$\int \frac{dx}{\sqrt{1+4p^2x^2}} = \frac{1}{2p} l(2px + \sqrt{1+4p^2x^2}),$$

$$\therefore s = \tfrac{1}{2}x(1 + 4p^2x^2)^{\frac{1}{2}} + \frac{1}{4p}l(2px + \sqrt{1 + 4p^2x^2}),$$

the origin of the integral being $x = 0$.

PROP. LXXXVI.

(270.) *To determine the arc of an hyperbolic curve represented by* $y = px^{-n}$.

The equation being differentiated, gives

$$dy = -npx^{-n-1}dx,$$

$$\therefore \sqrt{dy^2 + dx^2} = (1 + n^2p^2x^{-2n-2})^{\frac{1}{2}}dx,$$

$$\therefore ds = x^{-n-1}(x^{2n+2} + n^2p^2)^{\frac{1}{2}}dx,$$

$$\therefore s = \int x^{-n-1}(n^2p^2 + x^{2n+2})^{\frac{1}{2}} \cdot dx.$$

This does not come under the criterions of integration established in (221.), and can therefore only be obtained by approximation.

PROP. LXXXVII.

(271.) *To determine the arc of an ellipse.*

Let the equation $a^2y^2 + b^2x^2 = a^2b^2$ (see Geometry) be differentiated,

$$dy = -\frac{b^2x}{a^2y}dx,$$

$$\therefore dy^2 + dx^2 = \frac{a^4y^2 + b^4x^2}{a^4y^2}dx^2.$$

But $a^4y^2 + b^4x^2 = a^2b^2(a^2 - e^2x^2)$, and $a^4y^2 = a^2b^2(a^2 - x^2)$, where e represents the eccentricity. Hence

$$s = \int\frac{\sqrt{a^2 - e^2x^2}}{\sqrt{a^2 - x^2}}dx.$$

The series which gives the approximate value of this integral is given in (236.), Ex. 3.

If $x = a = 1$, and e be supposed very small, the series for the quadrant of the ellipse becomes

$$\frac{\pi}{2}\left(1 - \frac{1.1}{2.2}e^2 - \frac{1.1.1.3}{2.2.4.4}e^4 - \frac{1.1.1.3.3.5}{2.2.4.4.6.6}e^6 \cdot \cdot \cdot \cdot\right),$$

which gives the ratio of the circumference of the ellipse to that of a circle *quam proxime.*

PROP. LXXXVIII.

(272.) *To determine the area of a parabolic or hyperbolic curve represented by* $y = px^{\pm n}$.

Multiplying by dx, we find

$$da = px^{\pm n}dx,$$

$$\therefore a = p\frac{x^{\pm n+1}}{(\pm n+1)} = \frac{yx}{(\pm n+1)}.$$

If this integral be assumed between the limits yx and $y'x'$,

$$a = \frac{yx - y'x'}{n+1}.$$

This integration holds good in every case, except when $n = -1$, in which it becomes

$$a = pl\left(\frac{x}{x'}\right).$$

The integral taken between the limits x and x' being expressed thus,

$$a = p \cdot \frac{x^{n+1} - x'^{n+1}}{n+1},$$

shows that the area included between the entire curve and the axis of x can only be finite when $n + 1 < 0$, $\therefore n < -1$.

Thus the common hyperbola is the limit which divides the class of hyperbolas which intercept with their asymptotes finite areas from those which do not, and no parabola can include with its axis a finite area.

PROP. LXXXIX.

(273.) *To find the area included by two radii vectores from the centre of an equilateral hyperbola.*

The polar equation of this curve, related to the centre, is

$$r^2 \cos.2\omega = a^2.$$

Hence

$$\tfrac{1}{2}r^2 d\omega = \frac{a^2 d\omega}{2\cos.2\omega};$$

by Sect. X. Ex. 2,

$$\therefore \int \tfrac{1}{2}r^2 d\omega = \frac{a^2}{4} \cdot l \tan.(45^0 + \omega),$$

the origin of the integral being $\omega = 0$. If it be taken between the limits ω and ω',

$$\int \tfrac{1}{2}r^2 d\omega = \frac{a^2}{4} \cdot l \cdot \frac{\tan.(45^0 + \omega)}{\tan.(45^0 + \omega')}.$$

PROP. XC.

(274.) *To determine the surface and volume of a cylinder.*

A cylinder is produced by the revolution of a rectangle round one of its sides.

Hence, in the formula

$$a = 2\pi \int y dz,$$

y is constant, $\because$ $a = 2\pi y s$, y being the radius of the base of the cylinder, and s its altitude. Hence the surface of a

cylinder is found by multiplying its altitude into the circumference of its base.

For the volume

$$u = \pi\int y^2 dx, \because u = \pi y^2 x.$$

The volume is therefore found by multiplying the altitude by the area of the base.

PROP. XCI.

(275.) *To determine the surface and volume of a right cone.*

A right cone is a surface produced by the revolution of a rectilinear angle round one of its sides.

The vertex of the angle being assumed as origin, and the axis of rotation as axis of x, the equation of the generatrix is $y = px$, p being the tangent of the semiangle of the cone.

Hence, if a be its surface,

$$da = 2\pi\int y ds;$$

but $ds = \sqrt{1 + p^2} \cdot dx, \because$

$$a = 2\pi\sqrt{1 + p^2}\int px dx = \pi\sqrt{1 + p^2} \cdot px^2,$$

the origin of the integral being $x = 0$.

Or if s represent the side of the cone,

$$a = \pi ys.$$

Since πy is the semicircumference of the base, it appears that the surface of a right cone is equal to a triangle, whose altitude is equal to the side of the cone, and whose base equals the circumference of the base of the cone.

If the cone be truncated, the integral must be taken between the limits x and x' corresponding to the distances of its bases from the vertex. Hence

$$a = \pi\sqrt{1 + p^2} \cdot p(x^2 - x'^2);$$

but $(x - x')\sqrt{1+p^2} = s$, the side of the truncated cone. Hence

$$a = \pi(y + y')s.$$

Hence the surface of a truncated cone is equal to a trapezium, whose altitude is equal to the side, and whose parallel bases are equal to the circumferences of its bases; or it is equal to the sum of the surfaces of two cones, whose sides are equal to that of the truncated cone, and whose bases are equal to the two bases.

To find the volume (u),

$$u = \pi\int y^2dx = \pi\int p^2x^2dx,$$

$$\therefore u = \tfrac{1}{3}\pi p^2x^3 = \tfrac{1}{3}\pi y^2x,$$

the origin of the integral being $x = 0$.

Hence it appears that the volume of a right cone is found by multiplying its altitude x by one-third of its base πy^2, and that it is therefore one-third of a cylinder in the same base and altitude. (Euclid, lib. xii. prop. 10.) Hence also may easily be deduced, Euclid, lib. xii. props. 11, 12, 13, 14, 15.

If the cone be truncated,

$$u = \tfrac{1}{3}\pi p^2(x^3 - x'^3);$$

Since $x - x'$ is the altitude of the truncated cone, let it be A, $\therefore$

$$u = \tfrac{1}{3}\pi\text{A}(y^2 + yy' + y'^2).$$

The terms $\frac{1}{3}\pi\text{A}y^2$, $\frac{1}{3}\pi\text{A}y'^2$, are evidently the volumes of cones on the bases of the given truncated cone, and in the same altitude. And the term $\frac{1}{3}\pi\text{A}yy'$ is the volume of a cone in the same altitude, and having a base which is a mean proportional between the bases of the truncated cone. The given truncated cone is therefore equal to the sum of the volumes of these three cones.

PROP. XCII.

(276.) *Of the surface of a sphere.*

A circle being supposed to revolve on one of its diameters, generates a sphere. Let the equation of the generatrix be

$$y^2 + x^2 = r^2.$$

Differentiating, we find

$$dy = -\frac{xdx}{y},$$

$$\therefore dy^2 + dx^2 = \frac{y^2+x^2}{y^2}dx^2 = \frac{r^2dx^2}{y^2},$$

$$\therefore ds = \frac{rdx}{y},$$

$$\therefore a = 2\pi\int rdx = 2\pi r(x - x'),$$

the origin of the integral being x'.

If $x = r$, the formula expresses the volume of a spherical segment, whose base is a lesser circle of the sphere at the distance x' from the centre. Let that part of the *axis* of the segment (the diameter of the sphere passing through the centre of its base) intercepted within it be called v,

$$a = 2\pi r\text{v}.$$

It is evident that $2r\text{v}$ is equal to the square of the chord c of the arc, whose revolution generates the segment. Hence

$$a = \pi\text{c}^2.$$

The surface of the segment is therefore equal to the area of the circle described with this chord as radius. Hence the surface of an hemisphere is equal to the area of a circle described with a radius equal to the side of the square inscribed in a great circle, and the entire surface of the sphere is equal to the area of four great circles, or to the area of a circle described with a diameter of the sphere as radius.

The formula

$$a = 2\pi r(x - x')$$

expresses the surface of a cylinder, of which the altitude is $(x - x')$, and the radius r (274.). Hence it appears, that if a cylinder be circumscribed round a sphere, so that it will touch the sphere both with its sides and bases, the part of the cylindrical surface, intercepted between any two planes perpendicular to its axis, is equal to the part of the spherical surface intercepted by the same planes, and the whole surface of the sphere is equal to the cylindrical surface, exclusive of the bases of the cylinder. The spherical surface bears to the entire cylindrical surface, including the bases, the ratio 2 : 3.

If round the circle, whose revolution generates the sphere, an equilateral triangle be circumscribed, one of its vertices being on the axis of revolution, it will generate a cone, called an *equilateral* cone, from the circumstance of the diameter of its base being equal to its side. It appears from plane geometry, that the altitude of this cone will be $3r$, the radius of its base $\sqrt{3} . r$, and therefore its side $2\sqrt{3} . r$. The conical surface of this cone is, therefore, $6\pi r^2$, or equal to six times a great circle; and since its base is $3\pi r^2$, its whole surface is nine times a great circle. Since the circumscribed cylinder, including its bases, is six times a great circle, the three surfaces of the sphere, cylinder, and cone, are in geometrical progression, and in the ratio 2 : 3.

PROP. XCIII.

(277.) *Of the volume of a sphere.*

The formula

$$u = \pi \int y^2 dx$$

becomes by substituting for y^2 its value $r^2 - x^2$,

$$u = \pi\int(r^2 - x^2)dx,$$
$$\therefore u = \pi(r^2x - \tfrac{1}{3}x^3),$$

the origin of the integral being $x = 0$; or

$$u = \pi[r^2(x - x') - \tfrac{1}{3}(x^3 - x'^3)],$$
$$u = \pi(x - x')[r^2 - \tfrac{1}{3}(x^2 + xx' + x'^2)],$$

the origin being $x = x'$.

To determine the volume of a spherical segment, let $x = r$, $\therefore$

$$u = \tfrac{1}{3}\pi(r - x')[2r^2 - rx' - x'^2].$$

To extend the integral to the entire sphere, let $x' = -r$, $\therefore$

$$u = \tfrac{4}{3}\pi r^3,$$

which is the volume of a sphere, whose radius is r *.

Let a be the surface of the sphere. By the last proposition $a = 4\pi r^2$. Hence

$$u = \tfrac{1}{3}ar,$$

which is the volume of a cone whose altitude is r, and whose base is a. Hence the volume of a sphere is equal to that of a cone, having its base equal to the surface, and its altitude equal to the radius.

The volume of the circumscribed cylinder (274.) is $2\pi r^3$; since $2r$ is its altitude and πr^2 its base. Also the volume of the circumscribed cone is $3\pi r^3$, since its altitude is $3r$, and the radius of its base $\sqrt{3}\,.\,r$. Hence it appears that the volumes of the sphere, cylinder, and cone, as well as their surfaces, are in geometrical progression, and in the ratio $2 : 3$.

This beautiful property was the discovery of Archimedes, who was so charmed with it, that he is said to have ordered it to be engraved upon his tomb.

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* This formula evidently contains Euclid, lib. xii. prop. 18.

PROP. XCIV.

(278.) *To determine the volume of an ellipsoid.*

Let the equation of the ellipsoid be

$$a^2b^2z^2 + b^2c^2x^2 + a^2c^2y^2 = a^2b^2c^2.$$

The equation of a section perpendicular to the axis of z, and at a given distance z from the origin, is

$$a^2y^2 + b^2x^2 = \frac{a^2b^2(c^2-z^2)}{c^2}.$$

The semiaxes of this section are

$$\text{A} = \frac{a\sqrt{c^2-z^2}}{c},$$

$$\text{B} = \frac{b\sqrt{c^2-z^2}}{c}.$$

The area of the section is therefore (*Geometry*, 378.),

$$\text{AB}\pi = \frac{\pi ab(c^2-z^2)}{c^2}.$$

This being multiplied by dz, and the result integrated, gives

$$u = \frac{\pi ab}{c^2}(c^2z - \tfrac{1}{3}z^3),$$

$z = 0$ being the origin of the integral.

If the integral be taken between the limits z and z',

$$u = \frac{\pi ab}{c^2}(z - z')[c^2 - \tfrac{1}{3}(z^2 + zz' + z'^2)].$$

To determine the volume of a segment cut off by a plane at the distance z', let $z = c$, $\therefore$

$$u = \frac{\pi ab}{3c^2}(c - z')[2c^2 - z'^2 - cz'].$$

To extend the integral to the whole ellipsoid, let

$$z' = -c, \therefore$$

$$u = \tfrac{4}{3}\pi abc.$$

Hence the volume of the ellipsoid is equal to that of a sphere, the cube of whose radius is equal to the product of the semiaxes.

If the ellipsoid be generated by revolution round the axis a, $b = c$, and the volume is

$$u = \tfrac{4}{3}\pi ab^2.$$

If $a = b = c$, the formula gives the volume of a sphere, the same as was before obtained (277.).

SECTION XIV.

Of the integration of differentials of functions of several independent variables.

(279.) The differentials of functions of several variables are of two kinds, *partial* and *total* (94, 95.). The methods of integration are different for these. We shall first consider the integration of partial differentials.

As a partial differential is found by differentiating the primitive function, considering all the variables but one constant, so the integration must proceed upon the same hypothesis. To render the investigation more simple, we shall first consider functions of two variables only. The principles, when established, may be easily generalised. Let u be a function of x and y, and let the partial differential taken with respect to x be

$$\frac{du}{dx}dx = \text{N}dx.$$

In this, N is a function of x and y; but as it is derived from the function u by considering y constant, so in the integration, N is to be taken as a function of x only. Let the integral of Ndx, under this point of view, be U, ∴

$$u = \text{U} + \text{C},$$

C being an arbitrary constant. This, however, is only constant with respect to the variation of x, and is therefore to be considered as a function of y, let it be Y, $\therefore$

$$u = \text{U} + \text{Y}.$$

Hence it appears, that one partial differential is insufficient to determine the primitive function, but will determine that part of it which depends on the variable to which the partial differential is related.

In a similar way a partial differential of a superior order taken with respect to the same variable may be integrated by a series of successive integrations.

(280.) But when the partial differential of a superior order has been taken with respect to different variables, the process is different. Let M be a partial differential coefficient of the second order taken successively with respect to y and x. Then if u be the primitive function

$$\frac{d^2u}{dxdy} = \text{M}.$$

Let $\dfrac{du}{dx} = \text{U}$, $\because \dfrac{d^2u}{dxdy} = \dfrac{d\text{U}}{dy}$, $\therefore$

$$d\text{U} = \text{M}dy.$$

Integrating this, y alone being considered variable, and the arbitrary constant, which is a function of x, being X$'$, we find

$$\text{U} = \int \text{M}dy + \text{X}',$$

$$\therefore \frac{du}{dx} = \int \text{M}dy + \text{X}'.$$

Since the integral $\int \text{M}dy$ is known, let it be U$'$, $\therefore$

$$\frac{du}{dx}dx = \text{U}'dx + \text{X}'dx.$$

Let this be integrated, x only being considered variable, and we find

$$u = \int \text{U}'dx + \int \text{X}'dx + \text{Y},$$

Y being the arbitrary constant and a function of y.

If $\int \text{U}'dx = \text{U}$ and $\int \text{X}'dx = \text{X}$, $\because$

$$u = \text{U} + \text{X} + \text{Y},$$

As the value of $\frac{d^2u}{dxdy}$ is the same, whatever be the order in which the differentiations may have been performed, so the integral will be the same in whatever order the integrations may be performed. This is expressed analytically thus:

$$\int dx \int \text{M}dy = \int dy \int \text{M}dx.$$

Such an integral is therefore usually expressed

$$\iint \text{M}dxdy.$$

In a similar way, if u be a function of three variables, the integral of the differential

$$\frac{d^3u}{dxdydz}dxdydz = \text{M}dxdydz$$

may be obtained; but in this case there will be three arbitrary functions, and the integral will assume the form

$$\int \text{M}dxdydz = \text{U} + \text{X} + \text{Y} + \text{Z}.$$

And similar observations may be applied to differentials of superior orders.

(281.) As a total differential of a function of several variables is the sum of its partial differentials, so the integral of a total differential is the sum of the integrals of the partial differentials. If, therefore, the partial differentials be all given, the total differential may be found by the rules which have been established. In order, however, that the integration of a given total differential be possible, it will be necessary to ascertain whether the parts which compose it, involving the differentials of the variables respectively, are the several partial differentials of any *one* function of the variables; for this may not be the case, and if not, the formula is not the total differential of any function, and therefore cannot be integrated.

PROP. XCV.

(282.) *Given two functions of two variables, to determine whether they be partial differential coefficients of the same function, and if so, to find the primitive function.*

Let M and N be the two given functions of the variables x and y, and let the primitive function sought be u, so that we have

$$\frac{du}{dx} = \text{M}, \quad \frac{du}{dy} = \text{N}.$$

Each of these being integrated, give

$$\left.\begin{array}{l} u = \int \text{M}dx + \text{Y} \\ u = \int \text{N}dy + \text{X} \end{array}\right\}[1],$$

Y and X being arbitrary functions of y and x respectively.

If the two differential coefficients M and N be derivable from the *same* primitive function, it is necessary that these two values of u should be identical independently of the variables. Now since Y is independent of x, and X independent of y, it follows that Y must be identical with that part of the function $\int \text{N}dy$, which is independent of x, and X must be identical with that part of the function $\int \text{M}dx$, which is independent of y. These substitutions being made for X and Y, if the two values of u become perfectly identical, the two differential coefficients M and N must be derivable from the same primitive function, and that function is the common value of u thus found, an arbitrary constant being annexed. On this condition, therefore, and not otherwise, the differential

$$\text{M}dx + \text{N}dy$$

is capable of integration; and if this condition be not satisfied, the proposed differential is not the *exact differential* (a phrase implying an integrable differential) of any function.

(283.) The process for determining the functions X and Y may also be explained thus. Let the first of the equations [1] be differentiated for y, we find

$$\frac{du}{dy} = \frac{dV}{dy} + \frac{dY}{dy},$$

where $V = \int M dx$. Hence

$$Y = \int\left(\frac{du}{dy} - \frac{dV}{dy}\right)dy = \int\left(N - \frac{dV}{dy}\right)dy.$$

Hence the complete integral will be

$$u = \int M dx + \int\left(N - \frac{dV}{dy}\right)dy;$$

or otherwise by the second equation

$$u = \int N dy + \int\left(M - \frac{dV'}{dx}\right)dx$$

where $V' = \int N dy$.

(284.) The condition of integrability, already determined, may be otherwise expressed. It follows from what has been established, that if the two given partial differential coefficients be derivable from the same function, the formula

$$N - \frac{dV}{dy}$$

must be a function of y, and independent of x. Therefore, if it be differentiated for x, its differential coefficient must $= 0$, $\because$

$$\frac{dN}{dx} - \frac{d^2V}{dydx} = 0,$$

$$\therefore \frac{dN}{dx} = \frac{d^2V}{dydx} = \frac{d \cdot \frac{dV}{dx}}{dy}.$$

But $\frac{dV}{dx} = M$, $\therefore$

$$\frac{dN}{dx} = \frac{dM}{dy},$$

a condition which must be fulfilled, in order that the formula

$$\text{M}dx + \text{N}dy$$

should be a complete differential.

On the other hand, it is evident from the differential calculus, that if this be the complete differential of a function u, the above criterion must be fulfilled, for by (96.),

$$\frac{d^2u}{dxdy} = \frac{d^2u}{dydx},$$

$$\text{or } \frac{d\cdot\frac{du}{dx}}{dy} = \frac{d\cdot\frac{du}{dy}}{dx},$$

$$i.\ e.\ \frac{d\text{M}}{dy} = \frac{d\text{N}}{dx}.$$

This is usually called the *criterion of integrability.*

(285.) The theorem expressed by the formula

$$\frac{d^2\text{V}}{dxdy} = \frac{d\text{M}}{dy},$$

may be expressed also thus,

$$\frac{d\cdot\frac{d\text{V}}{dx}}{dy}\cdot dx = \frac{d\text{M}}{dy}dx.$$

By integrating, we find

$$\frac{d\text{V}}{dy} = \int\frac{d\text{M}}{dy}dx,$$

$$\therefore \frac{d\cdot\int\text{M}dx}{dy} = \int\frac{d\text{M}}{dy}dx,$$

which indicates a method of obtaining one partial differential coefficient of a function of two variables from the other, the arbitrary function being understood to be annexed.

(286.) The rules for the integration of differentials of several variables may be easily found by generalising those already given. Let M, N, L, be three functions, each being a function of x, y, and z; it is required to determine whe-

ther they be partial differential coefficients of the same function, and to determine that function; in other words, it is required to assign the conditions of integrability and the integral of

$$du = \text{M}dx + \text{N}dy + \text{L}dz,$$
$$\because u = \int(\text{M}dx + \text{N}dy + \text{L}dz).$$

Since M and N must be the partial differential coefficients of u, considered as a function of x and y, the conditions

$$\frac{d\text{M}}{dy} = \frac{d\text{N}}{dx},$$

must be fulfilled. In like manner it may be shown, that the conditions

$$\frac{d\text{M}}{dz} = \frac{d\text{L}}{dx},$$
$$\frac{d\text{N}}{dz} = \frac{d\text{L}}{dy},$$

must be also fulfilled. If the given differential coefficients fulfil these conditions, they must be derivable from the same function of x, y, z; for by the first, M and N are derivable from the same function of x, y; by the second, M and L are derivable from the same function of x, z; and by the last, N and L are derivable from the same function of y, z. Hence the three have the same integral.

It also follows, that

$$\text{M}dx + \text{N}dy,$$
$$\text{N}dy + \text{L}dz,$$
$$\text{L}dz + \text{M}dx,$$

are respectively exact differentials, and the integral of the proposed differential may be obtained by integrating any one of these, annexing an arbitrary function of the remaining variable. Thus the sought integral would be obtained under the forms

$$u = \text{U} + \text{Z},$$
$$u = \text{U}' + \text{X},$$
$$u = \text{U}'' + \text{Y},$$

Z, X, and Y being arbitrary functions of z, x, and y respectively. The function Z may be determined at once by substituting for it that part of the functions U′ or U″ which is independent of x or y; for since the values of u must be identical independently of the variables, those parts of them which are independent of x and y must be identical. In a similar manner, the arbitrary functions X and Y may be found.

These functions may also be determined thus. Let the first equation be differentiated for z. The result is

$$\mathrm{L}dz = \frac{d\mathrm{U}}{dz}dz + d\mathrm{Z},$$

$$\therefore \mathrm{Z} = \int\left(\mathrm{L} - \frac{d\mathrm{U}}{dz}\right) \cdot dz.$$

And, in like manner,

$$\mathrm{X} = \int\left(\mathrm{M} - \frac{d\mathrm{U}'}{dx}\right)dx,$$

$$\mathrm{Y} = \int\left(\mathrm{N} - \frac{d\mathrm{U}''}{dy}\right)dy.$$

The process for integrating differentials of any number of variables will now be evident. The number of equations which give the criterion of integrability is, in general, the number of different combinations of two variables, and is therefore $\frac{n \cdot \overline{n-1}}{1.2}$, n expressing the number of variables.

SECTION XV.

Praxis on the integration of differentials of several variables.

Ex. 1. Let $du = \frac{ydx - xdy}{y^2 + x^2}$. In this case,

$$\text{M} = \frac{y}{y^2+x^2}, \quad \text{N} = -\frac{x}{y^2+x^2},$$

$$\therefore \int \text{M}dx = \tan.^{-1}\frac{x}{y} + \text{Y},$$

$$\int \text{N}dy = \tan.^{-1}\frac{x}{y} + \text{X}.$$

Hence $\text{Y} = 0$ and $\text{X} = 0$, $\therefore u = \tan.^{-1}\dfrac{x}{y}$.

Ex. 2. Let

$$du = (3x^2 + 2axy)dx + (ax^2 + 3y^2)dy,$$

$$\therefore \text{M} = 3x^2 + 2axy, \quad \text{N} = ax^2 + 3y^2,$$

$$\therefore \int \text{M}dx = x^3 + ax^2y + \text{Y},$$

$$\int \text{N}dy = ax^2y + y^3 + \text{X}.$$

Hence $\text{Y} = y^3$ and $\text{X} = x^3$; and by these substitutions, the two integrals become identical. The differential is therefore integrable, and its integral is

$$u = x^3 + ax^2y + y^3.$$

Ex. 3. Let

$$du = (2\text{A}y + \text{B}x + \text{D})dy + (2\text{C}x + \text{B}y + \text{E})dx,$$

$$\therefore \text{M} = 2\text{C}x + \text{B}y + \text{E},$$

$$\text{N} = 2\text{A}y + \text{B}x + \text{D},$$

$$\therefore \int \text{M}dx = \text{C}x^2 + \text{B}yx + \text{E}x + \text{Y},$$

$$\int \text{N}dy = \text{A}y^2 + \text{B}yx + \text{D}y + \text{X}.$$

Hence $\text{Y} = \text{A}y^2 + \text{D}y$ and $\text{X} = \text{C}x^2 + \text{E}x$, by which alternate substitution, the formula becoming identical, proves that the differential is integrable, and that its integral is

$$u = \text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x.$$

Ex. 4. Let

$$du = \frac{dx}{y} + \frac{dy}{x} - \frac{ydx}{x^2} - \frac{xdy}{y^2}.$$

Hence

$$\text{M} = \frac{1}{y} - \frac{y}{x^2}, \quad \text{N} = \frac{1}{x} - \frac{x}{y^2},$$

$$\therefore \int \text{M}dx = \frac{x}{y} + \frac{y}{x} + \text{Y},$$

$$\int \mathrm{N}dy = \frac{y}{x} + \frac{x}{y} + \mathrm{X}.$$

Hence $\mathrm{X} = 0$ and $\mathrm{Y} = 0$, by which the equations becoming identical, the proposed differential is integrable, and its integral is

$$u = \frac{y}{x} + \frac{x}{y}.$$

Ex. 5. Let

$$du = xdy + ydx - \frac{dy}{xy^2} - \frac{dx}{yx^2},$$

$$\because \mathrm{M} = y - \frac{1}{yx^2}, \quad \mathrm{N} = x - \frac{1}{xy^2},$$

$$\because \int \mathrm{M}dx = yx + \frac{1}{yx} + \mathrm{Y},$$

$$\int \mathrm{N}dy = yx + \frac{1}{yx} + \mathrm{X}.$$

Hence $\mathrm{X} = 0$, $\mathrm{Y} = 0$, and

$$u = yx + \frac{1}{yx}.$$

Ex. 6. $du = \dfrac{2(ydx - xdy)}{y\sqrt{x^2 - y^2}}$. Hence

$$\mathrm{M} = \frac{2}{\sqrt{x^2 - y^2}}, \quad \mathrm{N} = -\frac{2x}{y\sqrt{x^2 - y^2}},$$

$$\because \int \mathrm{M}dx = 2l[x + \sqrt{x^2 - y^2}] + \mathrm{Y},$$

$$\int \mathrm{N}dy = 2l[x + \sqrt{x^2 - y^2}] - 2ly + \mathrm{X}.$$

Hence $\mathrm{X} = 0$ and $\mathrm{Y} = -2ly$, $\because$

$$u = 2l[x + \sqrt{x^2 - y^2}] - 2ly = 2l \cdot \frac{x + \sqrt{x^2 - y^2}}{y}.$$

Ex. 7. Let

$$u = \frac{dx}{x} + \frac{y^2 dx}{x^3} - \frac{ydy}{x^2} + \frac{(ydx - xdy)\sqrt{x^2 + y^2}}{x^3} + \frac{dy}{2y}.$$

Hence

$$\mathrm{M} = \frac{\sqrt{x^2 + y^2}(\sqrt{x^2 + y^2} + y)}{x^3},$$

$$\text{N} = \frac{1}{2y} - \frac{\sqrt{x^2+y^2}+y}{x^2},$$

$$\therefore \int \text{M}dx = \tfrac{1}{2}lx - \frac{y^2}{2x^2} - \frac{y\sqrt{x^2+y^2}}{2x^2} + \tfrac{1}{2}l[\sqrt{x^2+y^2}-y] + \text{Y},$$

$$\int \text{N}dy = \tfrac{1}{2}ly - \frac{y^2}{2x^2} - \frac{y\sqrt{x^2+y^2}}{2x^2} + \tfrac{1}{2}l[\sqrt{x^2+y^2}-y] + \text{X},$$

Hence $\text{Y} = \frac{1}{2}ly$ and $\text{X} = \frac{1}{2}lx$, and the other parts of these integrals being identical, the proposed differential is integrable, and its integral is

$$u = \tfrac{1}{2}l[xy(\sqrt{x^2+y^2}-y)] - \frac{y^2}{2x^2} - \frac{y\sqrt{x^2+y^2}}{2x^2}.$$

Ex. 8. Let

$$du = \frac{y^2-xy}{(x^2+y^2)^{\frac{3}{2}}}dx + \frac{x^2-xy}{(x^2+y^2)^{\frac{3}{2}}}dy,$$

$$\therefore \int \text{M}dx = \frac{x+y}{\sqrt{x^2+y^2}} + \text{Y},$$

$$\int \text{N}dy = \frac{x+y}{\sqrt{x^2+y^2}} + \text{X}.$$

Hence $\text{X} = 0$ and $\text{Y} = 0$, $\therefore$

$$u = \frac{x+y}{\sqrt{x^2+y^2}}.$$

SECTION XVI.

The general theory of differential equations and arbitrary constants.

(287.) Having in the preceding sections explained the methods of obtaining the integrals of differentials of one and of several variables, under all the varieties of form in which

they present themselves, we now come to the consideration of the methods of integrating *differential equations* *. As, however, this part of the science is of considerable importance and difficulty, before we enter upon the details of the methods, we shall offer some general observations on the nature of *differential equations*, on the connexion of differential equations of different orders with each other, and with the *primitive equation* or integral from whence they are derived, and on the constant quantities upon which that connexion depends.

(288.) If an equation between two variables x and y be differentiated, a differential equation will be obtained involving the quantities x, y, and $\frac{dy}{dx}$, the last occurring only in the first degree.

If this again be differentiated, an equation will be found involving x, y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$, the last, as before, entering it only in the first degree. In like manner the process may be continued and a series of differential equations obtained, each of which come under the form

$$\mathrm{A}\frac{d^ny}{dx^n} + \mathrm{B} = 0,$$

where A and B are, in general, functions of the variables, and the differential coefficients of orders inferior to the nth.

(289.) The order of a differential equation is determined by the highest differential coefficient which it contains, as the degree of an algebraic equation is determined by the

* The differential equations considered in this section are those between but two variables. Differential equations of several variables will be investigated in a subsequent section.

highest power of the unknown quantity. Thus, a differential equation, which contains no differential coefficient higher than the first, is said to be a differential equation of *the first order*. If it contain the second differential coefficient and none higher, it is called a differential equation of the second order, and so on.

(290.) Differential equations, like common algebraic equations, are also distinguished by *degrees*. These are marked by the highest power of the differential coefficient that marks their order, which enters them. Thus, a differential equation which involves no differential coefficient but the first, and that only in the first power, is called a differential equation of the *first order* and *first degree*. But if the differential coefficient enter in the second or third power, it is called a differential equation of the *first order* and *second degree* or *third degree*, and so on.

It appears from the process of differentiation, that no differential equation which is directly obtained from the primitive equation by differentiation alone can be of any degree but the first. Whenever, therefore, we meet a differential equation of a superior degree, it may at once be assumed not to be the immediate differential of any primitive equation. The origin of differential equations of superior degrees we shall find presently.

(291.) As an equation and its differential are deduced the one from the other, the same values of the variables which satisfy the former must also satisfy the latter. Hence it follows that other equations may be deduced by their combination. This circumstance indicates the existence of several differential equations of the same order depending upon the same primitive equation. Let $v = 0$ be the primitive equation between the variables x and y. By differentiating this, we obtain $v' = 0$, v' being a function of x, y, and $\frac{dy}{dx}$. In general, v' involves the same constant quan-

tities as v, except that constant of v which is independent of the variables x and y, for this disappears by differentiation. If v should not contain such a constant, its form is not general enough, and it is only a particular case of the integral of $v' = 0$. We shall, however, consider the integral $v = 0$ in its most general form, and shall therefore consider $v' = 0$ as containing all the constants of $v = 0$, except one. Thus, $v' = 0$ is the *immediate* differential equation of the first order derived from $v = 0$. Now if any one of the constants of $v' = 0$ be eliminated between the two equations, we shall obtain another differential equation of the first order between x, y, and $\frac{dy}{dx}$.

In this equation the constant which disappeared by differentiation will reappear, and another will disappear by elimination.

This latter differential equation will be perfectly distinct from the former, since a constant is involved in it which is excluded from the former, and since it excludes one which is involved in the former. The differential equation obtained by elimination may also differ in *degree* from that obtained by differentiation alone. If the constant which is eliminated enter the primitive equation in any dimensions higher than the first, this will necessarily be the case, as will presently appear. Hence the origin of differential equations of superior *degrees*.

A similar elimination may be practised upon each of the constants common to the two equations $v = 0$ and $v' = 0$, and as many different differential equations of the first order may be thus obtained as there are independent constants in the primitive equation.

(292.) If the differential coefficient be eliminated by any two of the differential equations of the first order, the result will be the primitive equation in which the two constants,

one of which is excluded from each of the differential equations, will appear.

Also, if either of the variables be eliminated by any two of these equations, the value of the differential coefficient will be obtained in terms of the other. In this case, also, if the variable eliminated exceed the first degree, the resulting differential equation will be of a superior degree also.

(293.) The several differential equations of the first order, all of which but one have been obtained by elimination, may also be obtained by differentiation *alone*, by slightly modifying the primitive equation. It has been shown that each of the differential equations of the first order excludes a constant of the primitive equation. In order to obtain the differential equation immediately by differentiation, let the primitive equation be supposed to be solved for the constant as if it were an unknown quantity, so that if A be the constant, the equation will assume the form $F(xy) - A = 0$. Under this form, the equation being differentiated, A will disappear, and an equation between the variables, the first differential coefficient, and all the other constants of the primitive equation will be found.

The equation thus obtained must be necessarily identical with, or reducible to, that obtained by elimination, since they involve the same variables and constants. In the same way all the differential equations of the first order which were before found by elimination, or their equivalents, may be immediately obtained by differentiation alone.

If the constant which is thus made to disappear by differentiation rise to the second or an higher degree in the primitive equation, then when the equation is solved it will have more values than one, and radicals will appear in the primitive equation which did not enter it before. These radicals will, therefore, also appear in the differential equation obtained from it, and therefore the differential coefficient

which must occur in the simple dimension will have several values. Now as this equation must be equivalent to that obtained by elimination, which does not include the above-mentioned radicals, it follows that they must be produced by solving it for the first differential coefficient, so as to reduce it to the same form as that obtained from mere differentiation. Hence it follows, that in this case the differential equation obtained by elimination must rise to the same degree as that of the constant in the primitive equation by whose elimination it was produced.

(294.) By differentiating the first differential equation $v' = 0$, the second differential equation $v'' = 0$ may be found. This will be the immediate differential equation of the second order of the proposed equation, but it will not be the only one. By what has been already observed of the first differential equation, it follows that the second differential equation $v'' = 0$ contains all the constants of the first $v' = 0$, except one, and therefore all the constants of the primitive equation $v = 0$, except two. The two which disappear by differentiation alone are those which are independent of the variables in the two equations $v = 0$ and $v' = 0$. A differential equation may, however, be obtained independent of any two constants A and B of the primitive equation, and may be obtained from two, and only two, of the differential equations of the first order.

1°. By differentiating the equation of the first order which excludes the constant A, and by it and its differential eliminating B, a differential equation, independent of A and B, will result; or the same may be obtained by solving the equation for the constant B, and then differentiating it (293.).

2°. By differentiating the equation of the first order which is independent of the constant B, and then eliminating A, or, as before, first solving the equation for A, and then differentiating the result, the same equation as before will be

obtained. Thus, this equation may be considered as a common differential of the two equations of the first order, the one independent of A, and the other of B.

(295.) In this way any two constants of $V = 0$ may be eliminated; and, therefore, there are as many different equations of the second order derived from the same primitive equation as there are different combinations of two constants in the original equation $V = 0$. If n be the number of constants, therefore, $\frac{n.n-1}{1.2}$ will be the number of different differential equations of the second order, each of which may be considered as a common differential of the two equations of the first order, which are independent severally of the two constants which are excluded from it. It is evident that these equations are all perfectly distinct, since they differ in their constants.

In like manner there may be $\frac{n.n-1.n-2}{1.2.3}$ differential equations of the third order derived from the primitive equation $V = 0$, each of them excluding three constants of the primitive equation. Each of these may indifferently be derived from three of the differential equations of the second order, scil. those three which exclude severally the three pairs of constants which may be combined from the three constants excluded from the differential equation of the third order.

These several differential equations of the third order may be obtained, either by obtaining one by differentiation, and the others by eliminating successively the constants between that and the equation of the second order; or they may be obtained without elimination by solving the differential equations of the second order for the constants successively, and then differentiating.

(296.) By continuing this reasoning, it follows,

1°. That in a differential equation of the mth order there are a number of constants equal to $n - m$, n being, as before, the number in the original equation.

2°. That a differential equation of the mth order may always be obtained either by elimination united with differentiation, by which any combination of m constants shall be excluded, or by successively solving the equations for the constants and differentiating.

3°. That therefore there will be

$$\frac{n.n-1.n-2.....n-(m-1)}{1.2.3.....m}$$

differential equations of the mth order derived from the same primitive equation, perfectly distinct from one another, since they differ in their constants.

4°. That each of these differential equations may be derived indifferently by differentiating m of the differential equations of the $(m-1)$th order, scil. those which exclude the m combinations of $(m-1)$ of the constants excluded from the differential equation of the mth order.

5°. That the differential equations of any order obtained by differentiation alone are always of the *first degree* with respect to the differential coefficient which marks their *order*, while those which are obtained by elimination are of the *same degree* as the constant by whose elimination they were obtained. The two equations will become identical by solving the latter for the differential coefficient.

6°. That if by two different differential equations of the mth order the mth differential coefficient be eliminated, a differential equation of the $(m-1)$th order will be obtained, including one constant more than either of those from which it was deduced, and therefore only excluding $(m-1)$ constants of the primitive equation, and this equation must therefore be identical with that differential equation deduced by differentiation and elimination, which includes the same

constants. It is evident that this elimination may be continued upwards until we arrive at the primitive equation.

7°. A differential equation of the nth order will include no constant, since, in that case, the number of constants eliminated is n. There will also be but one differential equation of this order, since, in this case, the formula expressing the number of differential equations becomes

$$\frac{n.\overline{n-1}.\overline{n-2}\ldots\overline{n-(n-1)}}{1.2.3\ldots n} = 1.$$

(297.) The conclusions at which we have just arrived resulted from the consideration of the process by which the several orders of differential equations are derived from a primitive equation between two variables. Let us now consider what these results suggest in returning upon our steps and ascending through the differential equations of the several orders to their original or primitive equation.

(298.) As by differentiating an equation, a constant disappears, so it should reappear upon integrating; and as only one constant can be removed by one differentiation, so one only should be introduced by one integration. The value of the constant introduced in any integration cannot be determined by the differential equation *alone*, since a differential equation is the same, whatever be the value of the constant which has been eliminated. Hence, as far as the differential equation is concerned, this constant is arbitrary, and any value whatever may be ascribed to it. In ascending, therefore, from a differential equation of the first order to its primitive or *integral*, one arbitrary constant, and but one, ought to be introduced, otherwise, the integral which will be obtained will not have all the generality which it ought to have.

(299.) If two different differential equations of the first order derived from the same primitive equation be given, the integration may be effected by eliminating the first dif-

ferential coefficient between them; the resulting equation between the two variables, and independent of differentials, will be the sought integral.

(300.) As a differential equation of the second order is immediately obtained from one of the first order, and is related to it in the same way as that of the first order is related to the primitive equation, it follows from what has been said, that the *first integral* of a differential equation of the second order is a differential equation of the first order, and that one, and only one arbitrary constant must be introduced in the integral. The *primitive absolute equation*, or *final* integral, is to be obtained by the integration of the differential equation of the first order thus obtained, in which integration a second arbitrary constant must appear. There is, however, another method of ascending to the final integral.

Since each differential equation of the second order may be indifferently derived from two of the first order, it follows that a differential equation of this kind has two *first integrals*. If both of these can be obtained, each including an arbitrary constant, the primitive absolute equation, or final integral, may be obtained by eliminating the first differential coefficient between them.

(301.) This principle of differential equations of the second order admitting two integrals, also furnishes a method of integrating differential equations of the first order. If an equation of the first order be differentiated, and thence one of the second order obtained, this admitting of another integral different from that from which it was derived by differentiation, this other integral may be found by integrating and introducing an arbitrary constant. Thus two differential equations of the first order will be obtained involving one arbitrary constant; by these the differential coefficient being eliminated, the final integral, including an arbitrary

constant, will be found. This is frequently the easiest method of integrating an equation of the first order and any degree superior to the first.

(302.) In like manner the first integral of a differential equation of the third order is a differential equation of the second order, including one arbitrary constant, and each differential equation of the third order has three different integrals of the second order. And, in general, the first integral of a differential equation of the mth order is a differential equation of the $(m-1)$th order; and each differential equation of the mth order admits m different first integrals, which are all differential equations of the $(m-1)$th order, and include m different arbitrary constants. If these m first integrals be obtained, the final integral may be found by mere elimination without further integration. For the m differential equations of the $(m-1)$th order include in general $(m-1)$ differential coefficients, scil. all the differential coefficients from the first to the $(m-1)$th order inclusive. These $(m-1)$ quantities may be eliminated by the m equations, and the result will be an equation independent of differentials including m arbitrary constants. This is the final integral in its most general state.

(303.) The integration of a differential equation of the first order may be effected by deducing from it by successive differentiation a differential equation of the mth order. If a first integral of this can be obtained different from the differential equation of the $(m-1)$th order from which it was derived, and including an arbitrary constant, the final integral can thence be obtained by elimination alone; for there are the differential equations from the first to the $(m-1)$th order inclusive obtained by differentiation, and also another of the $(m-1)$th order obtained by integration, making in all m equations to eliminate $(m-1)$

differential coefficients. The result being an equation free from differentials, and including one arbitrary constant, will be the integral of the proposed equation.

(304.) In the preceding observations we have assumed two propositions, 1°. That the final integral or primitive absolute equation, of a differential equation of the mth order, should include m arbitrary constants, in order to have all the generality which is due to it; and 2°. That a differential equation of the mth order admits of m different first integrals. Although these propositions seem sufficiently evident by retracing the process of differentiation, yet, as it is desirable to give to the theory established in the present section all the perfection and rigour possible, we shall subjoin direct demonstrations of these two principles.

PROP. XCVI.

(305.) *Every differential equation between two variables has an integral, and the integral of a differential equation of the* m*th order must, if in its most general state, include* m *arbitrary constants, and no more.*

The differential equation of the mth order determines the mth differential coefficient A_m as a function of the variables and the differential coefficients

$$A_1, A_2, \dots A_{m-1},$$

of the inferior orders.

By successive differentiation the differential equations of the superior orders may be found, and these will therefore be also determined as functions of the variables and of the differential coefficients of orders inferior to the mth; for the differential coefficients of the intermediate orders may be successively eliminated.

By Taylor's series we have

$$y' = y + A_1\frac{h}{1} + A_2\frac{h^2}{1.2} + A_3\frac{h^3}{1.2.3} \cdot \cdot \cdot \cdot$$

Let b be any value of x which does not render any of the differential coefficients derivable from the primitive equation infinite (55), and let $h = x - b$.

Let b be supposed to be substituted for x in the functions

$$y, \; A_1, \; A_2, \; \cdot \cdot \cdot \cdot$$

so that they will become constant quantities,

$$a_0, \; a_1, \; a_2, \; \cdot \cdot \cdot \cdot$$

and let the value of y' corresponding to x be y.

Thus the series becomes

$$y = a_0 + a_1 \cdot \frac{x-b}{1} + a_2 \cdot \frac{(x-b)^2}{1.2} + a_3 \cdot \frac{(x-b)^3}{1.2.3} + \cdot \cdot \cdot \cdot$$

the coefficients of which are all constant. The coefficients of this series from the $(m + 1)$th term forward are given functions of the coefficients of the first m terms, since they are what

$$A_m, \; A_{m+1}, \; A_{m+2}, \; \cdot \cdot \cdot \cdot$$

become when $x = b$; but these are determinate functions of

$$y, \; A_1, \; A_2, \; A_3, \; \cdot \cdot \cdot \cdot$$

and, therefore,

$$a_m, \; a_{m+1}, \; a_{m+2}, \; a_{m+3}, \; \cdot \cdot \cdot \cdot$$

are determinate functions of

$$a_0, \; a_1, \; a_2, \; a_3, \; \cdot \cdot \cdot \cdot$$

The series expressing the value of y is therefore the integral of the proposed equation, and contains m arbitrary constants, scil.

$$a_0, \; a_1, \; a_2, \; \cdot \cdot \cdot \; a_{m-1},$$

and no more.

This series is the development of the value of y in the final integral of the proposed equation, and may therefore represent that integral.

PROP. XCVII.

(306.) *Every differential equation of the* m*th order has* m *different first integrals, which are differential equations of the* (m — 1)*th order.*

The final integral gives y as an implicit function of x. Let it be expressed as an explicit function of x, so that $y = \text{F}(x)$. By Taylor's series,

$$\text{F}(x+h) = y + \text{A}_1\frac{h}{1} + \text{A}_2\frac{h^2}{1.2} + \text{A}_3\frac{h^3}{1.2.3}\cdots$$

If h be supposed to become $= -x$, $\because x + h = 0$, and therefore $\text{F}(x+h)$ becomes what the value of y is when $x = 0$. Let this value be y^0, $\because$

$$y^0 = y - \text{A}_1\frac{x}{1} + \text{A}_2\frac{x^2}{1.2} - \text{A}^3\frac{x^3}{1.2.3} + \cdots\cdots [1].$$

In the same manner, by successively considering A_1, A_2, A_3, $\cdots$ functions of x, we obtain

$$\text{A}_1^0 = \text{A}_1 - \text{A}_2\frac{x}{1} + \text{A}_3\frac{x^2}{1.2} - \cdots\cdots [2],$$

$$\text{A}_2^0 = \text{A}_2 - \text{A}_3\frac{x}{1} + \text{A}_4\frac{x^2}{1.2} - \cdots\cdots [3],$$

$$\text{A}_3^0 = \text{A}_3 - \text{A}_4\frac{x}{1} + \text{A}_5\frac{x^2}{1.2} - \cdots\cdots [4].$$

. . . .

. . . .

If a differential equation of the first order be given, it will determine the first and all the succeeding differential coefficients as functions of the variables. In this case the equation [1] will represent the primitive equation involving but one arbitrary constant y^0.

If a differential equation of the second order be given, it will determine the second differential coefficient and all the

coefficients of superior orders as functions of the variables and the first differential coefficient. In this case [1] and [2] represent two integrals of this equation, each including an arbitrary constant y^0 and A_1^0; all the other terms being functions of the variables and the first differential coefficient.

In like manner, if a differential equation of the third order be given, the three equations [1], [2], [3], represent its first integrals, each involving an arbitrary constant y^0, A_1^0, A_2^0, and being differential equations of the second order, and so on.

SECTION XVII.

Of the integration of differential equations of the first order and first degree, in which the variables are separable.

(307.) As the rules for differentiating functions of two variables equally apply to equations of two variables, so also the rules for integrating differentials of two variables apply to the integration of equations of two variables; and as there are many differentials of two variables which are not *exact differentials*, so also there are many differential equations which are not the immediate differentials of any primitive equation, and which are not therefore immediately integrable.

When a differential equation has been reduced to the form

$$\text{M}dx + \text{N}dy = 0,$$

its immediate integrability may be ascertained by the criterion (284.), and its integral found by the rules already established for differentials of two variables.

(308.) But although an equation may not come under the criterion of integrability of *functions* of two variables, we are not therefore to conclude that it is not integrable. We may, indeed, pronounce it at once not to be the *immediate* differential of any equation between the variables, because, if it were, it must come under the criterion. But it may be one of those differential equations which are not obtained by mere differentiation, but by eliminating some constant between the primitive equation and its immediate differential; or it may have happened, that some function of the variables having been a factor of the immediate differential, it was expunged after differentiation. Thus, for example, if the differential equation of a given equation between x and y were

$$(y^2 + x^2)\text{F}(xy)dy + (y^2 + x^2)\text{F}'(xy)dx = 0;$$

we should immediately expunge the common factor y^2+x^2; and although the above equation would come under the criterion of integrability, yet, after division by $(y^2 + x^2)$, it might no longer come under it. Thus, though the criterion applies to differentials, yet it does not to differential equations; at least, it does not apply as a *criterion*, properly so called. Because, although every equation which comes under the criterion can be immediately integrated, yet we cannot infer the converse, as has been shown.

(309.) Various analytical contrivances have been therefore invented for rendering integrable differential equations which do not fulfil the criterion of integrability. One of the most simple, when it can be effected, is the *separation of the variables*, or the reduction of the equation to the form

$$\text{X}dx + \text{Y}dy = 0.$$

In which state it is immediately integrable by the rules for integrating differentials of a single variable, the integral being

$$\int \text{X}dx + \int \text{Y}dy = 0\,*.$$

(310.) In differential equations of this kind, the variables are said to be *separated,* and therefore all equations in which such a *separation* can be effected may be considered as integrable by the above method. The most remarkable classes of equations, in which this can be effected, are the following:

1°. All differential equations coming under the form

$$\text{X}dy + \text{Y}dx = 0.$$

2°. All differential equations of the form

$$\text{XY}dy + \text{X}'\text{Y}'dx = 0.$$

3°. All homogeneous equations; that is, all algebraic equations in which the sum of the dimensions of x and y in every term is the same, and which, therefore, come under the form

$$\text{F}(x^m, x^{m-1}y, x^{m-2}y^2 \cdots)dy + \text{F}'(x^m, x^{m-1}y, x^{m-2}y^2 \cdots)dx = 0.$$

4°. *Linear equations;* that is, equations which involve y and dy only in the simple dimension, and which, therefore, come under the form

$$dy + (\text{X}y + \text{X}')dx = 0.$$

5°. The equation of *Riccati* (an Italian mathematician),

$$dy + (\text{A}y^2 + \text{B}x^m)dx = 0,$$

in which, in certain cases, the variables may be separated.

There are other equations in which the variables may be

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* This would be, according to the usual custom, expressed $\int \text{X}dx + \int \text{Y}dy = \text{C}$, C being an arbitrary constant. This, however, I conceive superfluous, if not positively wrong, since the introduction of the constant is a part of the operation indicated by the sign $\int$. I have, therefore, generally neglected the constant, except where the integration has been actually effected; then it is proper and necessary to introduce it, because the symbol which implies its introduction has disappeared.

separated, but these will sufficiently illustrate the principle. It is evident that all equations which, by any transformation, may be reduced to any of the preceding forms, may be integrated in the same manner.

(311.) 1°. The equation

$$\mathrm{X}dy + \mathrm{Y}dx = 0$$

being divided by XY, is reduced to

$$\frac{dy}{\mathrm{Y}} + \frac{dx}{\mathrm{X}} = 0,$$

the integral of which is immediately obtained,

$$\int \frac{dy}{\mathrm{Y}} + \int \frac{dx}{\mathrm{X}} = 0.$$

(312.) 2°. The equation

$$\mathrm{XY}dy + \mathrm{X'Y'}dx = 0$$

being divided by XY′, becomes

$$\frac{\mathrm{Y}}{\mathrm{Y'}}dy + \frac{\mathrm{X'}}{\mathrm{X}}dx = 0,$$

which is immediately integrable,

$$\int \frac{\mathrm{Y}}{\mathrm{Y'}}dy + \int \frac{\mathrm{X'}}{\mathrm{X}}dx = 0.$$

(313.) 3°. Each term of an homogeneous equation being of the form $\mathrm{A}y^n x^{m-n}$, the constant sum of the exponents being m; if every term of the equation be divided by x^m, the form of each term will become $\mathrm{A}\left(\frac{y}{x}\right)^n$. If $\frac{y}{x} = z$, the equation will assume the form

$$\mathrm{F}(z)dy + \mathrm{F'}(z)dx = 0.$$

But since $y = xz$, ∴ $dy = xdz + zdx$. Which being substituted for y, gives

$$x\mathrm{F}(z)dz + [\mathrm{F'}(z) + z\mathrm{F}(z)]dx = 0,$$

$$\therefore \frac{\mathrm{F}(z)}{\mathrm{F'}(z) + z\mathrm{F}(z)}dz + \frac{dx}{x} = 0,$$

$$\therefore \frac{1}{\frac{F'(z)}{F(z)}+z}dz+\frac{dx}{x}=0,$$

which is of the form

$$zdz + \text{X}dx = 0,$$

in which the variables are separated.

Equations are frequently rendered homogeneous by substituting for x and y, $x'+a$ and $y'+b$, and disposing of the arbitrary quantities a and b, so as to take out the terms which destroy the homogeneity, changing dx and dy iuto dx' and dy'. The analyst, however, must be determined in the choice of a fit transformation by the nature of the equation in each particular case.

(314.) 4°. In the linear equation

$$dy + (\text{X}y + \text{X}')dx = 0.$$

Let $\text{X}''z = y$, $\therefore dy = \text{X}''dz + zd\text{X}''$, by which substitutions the proposed equation becomes

$$\text{X}''dz + zd\text{X}'' + \text{X}\text{X}''zdx + \text{X}'dx = 0,$$

in which X'' is an arbitrary function of x. Let X'' be such as to fulfil the condition

$$zd\text{X}'' + \text{X}'dx = 0,$$

$$\therefore dz + \text{X}zdx = 0,$$

$$\therefore \frac{dz}{z} = -\text{X}dx, \quad \therefore z = e^{-\int \text{X}dx}.$$

Hence we find

$$d\text{X}'' = -e^{\int \text{X}dx}\text{X}'dx,$$

$$\therefore \text{X}'' = -\int e^{\int \text{X}dx}\text{X}'dx,$$

$$\therefore y = -e^{-\int \text{X}dx}\int e^{\int \text{X}dx}\text{X}'dx.$$

(315.) 5°. In the equation of Riccati,

$$dy + (\text{A}y^2 + \text{B}x^m)dx = 0;$$

if $m=0$, it becomes

$$dx + \frac{dy}{\text{A}y^2+\text{B}} = 0,$$

in which the variables are separated.

But if m be not $= 0$, let

$$y = \frac{z}{x^2} + \frac{1}{\text{A}x},$$

$$\therefore dy = \frac{xdz - 2zdx}{x^3} - \frac{dx}{\text{A}x^2}.$$

By which substitutions the given equation becomes

$$x^2dz + \text{A}z^2dx + \text{B}x^{m+4}dx = 0.$$

If in this $m = -2$, it is homogeneous; and if $m = -4$, the variables may be immediately separated by dividing the whole equation by $x^2(\text{A}z^2 + \text{B})$.

If, however, m be not $= -2$, nor $= -4$, a further transformation must be effected. Let

$$z = \frac{1}{t}, \quad x^{m+3} = u;$$

and let

$$n = -\frac{m+4}{m+3}, \quad \text{A}' = \frac{-\text{B}}{m+3},$$

$$\text{B}' = -\frac{\text{A}}{m+3}.$$

We find by these substitutions, that the equation becomes

$$dt + (\text{A}'t^2 + \text{B}'u^n)du = 0;$$

this being similar to the first equation, can be integrated when $n = -2$, or $n = -4$.

If n be not $= -2$, nor $= -4$, by continually repeating the same transformation, the equation may successively be reduced to a series of equations of the same form as the given one, and in which the exponent of the variable becomes successively equal to

$$-\frac{m+4}{m+3}, \; -\frac{n+4}{n+3}, \; -\frac{p+4}{p+3},$$

$$\text{or } -\frac{m+4}{m+3}, \; -\frac{3m+8}{2m+5}, \; -\frac{5m+12}{3m+7}, \; -\frac{7m+16}{4m+9} \ldots$$

The equation can only be integrated by the methods above

given, when some one of these is either $= 0$, $= -2$, or $= -4$, that is, when m is a number coming under the formula

$$\frac{-4n}{2n-1},$$

n being any positive integer, or $= 0$.

If the transformation $y = \frac{1}{t}$, $x^{m+1} = z$ had been made in the given equation, the same process would show that the integration could be effected when $m = \frac{-4n}{2n+1}$. The criterion of the integrability of the given equation by this method is then $m = \frac{-4n}{2n \pm 1}$, n being a positive integer, or $= 0$.

SECTION XVIII.

On the multipliers which render differential equations integrable.

(316.) In order that a differential equation of any order should be immediately integrable, it is necessary that it should be of the first degree with respect to the differential coefficient, which marks its order (290.). Otherwise, it has been the result, not of differentiation, but of elimination. But if it be of the first degree, it may always be considered as proceeding from the immediate differentiation of the differential equation of the next degree inferior to it, solved for the constant which has been eliminated.

(317.) Let a differential equation of the mth order be then supposed to be reduced to the form

$$\frac{d^m y}{dx^m} + \text{v} = 0 \quad \cdot \cdot \cdot \cdot \cdot \quad [1],$$

where U is in general a function of the variables and the differential coefficients of the inferior orders.

Let the differential equation, from which this is conceived to have been derived, be

$$U' = a \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot [2],$$

a being supposed to be the constant which has disappeared by differentiation, and U' being a function of the variables and the differential coefficients of orders inferior to the mth. This being differentiated, gives an equation of the form

$$u \cdot \frac{d^m y}{dx^m} + u' = 0 \cdot \cdot \cdot \cdot [3],$$

u and u' being likewise functions of the variables and the differential coefficients of orders inferior to the mth. Since this equation must be equivalent to the first, we have

$$U = \frac{u'}{u} \therefore Uu = u'.$$

Hence the latter equation becomes

$$u \cdot \frac{d^m y}{dx^m} + uU = 0,$$

which is an immediate differential, and therefore integrable. But this is the given equation [1] multiplied by the function u.

(318.) This multiplier is not the only one which will render the equation integrable. Let the equation [2] be multiplied by any function of a. This function being constant, the equation [3] becomes

$$F(a)u\frac{d^m y}{dx^m} + F(a)u' = 0,$$

$$\text{or } F(a)u\frac{d^m y}{dx^m} + F(a)uU = 0;$$

but by [2] this becomes

$$F(U')u\frac{d^m y}{dx^m} + F(U')uU = 0,$$

which is the exact differential of the equation

$$U'F(U') = aF(U').$$

But it is the equation [1] multiplied by $F(U')u$. Since the function $F(U')$ is arbitrary, there are an infinite variety of multipliers which will render the proposed equation integrable, scil. all those of which, u being one factor, the other is any function of U'.

(319.) Since by (306.) a differential equation of the mth order has m different first integrals, we may obtain a class of multipliers from each of them, which will render integrable the proposed equation of the mth order.

(320.) Having explained the general principle, we shall now apply it to differential equations of the first order and first degree. Let

$$Mdx + Ndy = 0$$

be the proposed equation, of which the primitive or integral is $U' = a$. By comparing this with the general formula already established (317.), we find $U = \frac{M}{N}$. The equation is rendered integrable by multiplying it by $uF(U')$. First, suppose $F'(U) = 1$, the equation becomes

$$uMdx + uNdy = 0.$$

Subjecting this to the criterion of integrability (284.), we find

$$\frac{d(Mu)}{dy} = \frac{d(Nu)}{dx},$$

$$\therefore u\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = N\frac{du}{dx} - M\frac{du}{dy}.$$

Since M and N are supposed to be given functions of x and y, this equation, when integrated and solved for u, would determine its value. It being, however, an equation of partial differentials, its solution can very seldom be effected; and even when it can, it presents generally greater difficulties than the proposed equation.

(321.) Although we cannot therefore in general determine a factor which will render an equation integrable; yet there are some properties of these factors which merit attention.

1°. If the integral of the differential

$$u\text{M}dx + u\text{N}dy$$

were known, the factor u could be found; for the above formula is identical with

$$\frac{du}{dx}dx + \frac{du}{dy}dy,$$

therefore we should be able to deduce the value of u by comparing them.

2°. If the factor u were known, an infinite number of other factors which would render the equation integrable could be found, as has been already shown.

3°. The factor u may, in some cases, be a function of one of the variables only. It may be easily discovered whether this be the case, and if it be found so, the factor u may be determined. Let u be supposed to be a function of the variable x. If so, $\frac{du}{dy} = 0$, ∴

$$u\left\{\frac{d\text{M}}{dy} - \frac{d\text{N}}{dx}\right\} = \text{N}\cdot\frac{du}{dx},$$

$$\therefore \frac{du}{u} = \frac{dx}{\text{N}}\left(\frac{d\text{M}}{dy} - \frac{d\text{N}}{dx}\right).$$

If the second member of this equation be independent of y, then u is a function of x alone, and not otherwise. Since M and N are given functions, this can always be determined. If it be so, the value of u is determined by the equation

$$lu = \int \frac{1}{\text{N}}\left(\frac{d\text{M}}{dy}dx - \frac{d\text{N}}{dx}dx\right) = \int \text{X}dx,$$

$$\therefore u = e^{\int \text{X}dx}.$$

(322.) Homogeneous functions have a remarkable property, which enables us to assign the factor which renders an homogeneous equation integrable. To explain this pro-

perty, let u represent an homogeneous function of x and y. In u, let x be changed into $x(1+h)$, and y into $y(1+h)$, and let the function become u', so that

$$u = \mathrm{F}(xy), \quad u' = \mathrm{F}(x + hx, y + hy).$$

Since u is an homogeneous function, $u' = (1+h)^m u$, m being the number of dimensions of the variables in each term of u, let these two values of u' be developed, the one in powers of hx and hy by (96.), the other in powers of h by the binomial theorem. Hence

$$u' = u + \frac{du}{dx}\cdot hx + \frac{du}{dy}\cdot hy + \frac{d^2u}{dx^2}\cdot\frac{h^2x^2}{1.2} + \frac{d^2u}{dxdy}\cdot\frac{h^2xy}{1} + \frac{d^2u}{dy^2}\cdot\frac{h^2y^2}{1.2} + \cdots$$

$$u' = u(1 + mh + \frac{m\cdot m-1}{1.2}h^2 + \frac{m\cdot m-1\cdot m-2}{1.2.3}h^3 \ldots).$$

Hence, by equating the corresponding coefficients, we find

$$mu = \frac{du}{dx}x + \frac{du}{dy}y,$$

$$\frac{m.m-1}{1.2}\cdot u = \frac{d^2u}{dx^2}\ \frac{x^2}{1.2} + \frac{d^2u}{dxdy}\ \frac{xy}{1} + \frac{d^2u}{dy^2}\ \frac{y^2}{1.2}$$

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. . . .

It is evident that this property belongs to homogeneous functions of any number of variables.

(323.) Let the equation to be integrated be

$$\mathrm{M}dx + \mathrm{N}dy = 0,$$

where M and N represent homogeneous functions of the variables. Let u be the sought factor and also an homogeneous function, and let it be supposed that

$$\mathrm{M}udx + \mathrm{N}udy = 0$$

is an exact differential. Hence

$$\frac{d(\mathrm{M}u)}{dy} = \frac{d(\mathrm{N}u)}{dx}.$$

Let the dimensions of M be p, and those of u, n, $\therefore$

$$(p+n)\mathrm{M}u = \frac{d(\mathrm{M}u)}{dx}x + \frac{d(\mathrm{M}u)}{dy}y,$$

$$\therefore (p+n)\text{M}u = \frac{d(\text{M}u)}{dx}x + \frac{d(\text{N}u)}{dx}y,$$

$$\therefore (p+n)\text{M}u = \frac{d(\text{M}ux + \text{N}uy)}{dx} - \text{M}u,$$

$$\therefore (p+n+1)\text{M}u = \frac{d\cdot u(\text{M}x+\text{N}y)}{dx}.$$

This equation is fulfilled by the conditions

$$p = -(n+1),$$

$$u = \frac{1}{\text{M}x+\text{N}y}.$$

Hence the equation

$$\frac{\text{M}dx + \text{N}dy}{\text{M}x + \text{N}y} = 0$$

is integrable.

SECTION XIX.

Praxis on the integration of differential equations of the first order and first degree.

I.

Differential equations which answer the criterion of integrability.

Ex. 1. $(2axy - y^3)dx + (ax^2 - 3xy^2)dy = 0,$

$$\text{M} = 2axy - y^3, \quad \text{N} = ax^2 - 3xy^2,$$

$$\int \text{M}dx = ax^2y - y^3x + \text{Y},$$

$$\int \text{N}dy = ax^2y - xy^3 + \text{X}.$$

Hence $\text{Y} = \text{X} = 0$, and the sought integral is

$$ax^2y - y^3x = \text{C}.$$

Ex. 2. $\frac{dx}{\sqrt{1+x^2}} + adx + 2bydy = 0,$

$$\because \text{M} = \frac{1}{\sqrt{1+x^2}} + a, \quad \text{N} = 2by,$$

$$\int \text{M}dx = ax + l(x + \sqrt{1+x^2}) + \text{Y},$$
$$\int \text{N}dy = by^2 + \text{X}.$$

Hence

$$\text{X} = ax + l(x + \sqrt{1+x^2}),$$
$$\text{Y} = by^2.$$

Therefore the sought equation is

$$by^2 + ax + l(x + \sqrt{1+x^2}) + \text{C} = 0,$$

C being the arbitrary constant.

Ex. 3. $\frac{a(xdx+ydy)}{\sqrt{y^2+x^2}} + \frac{ydx-xdy}{y^2+x^2} + 3by^2dy = 0,$

$$\because \text{M} = \frac{ax}{\sqrt{y^2+x^2}} + \frac{y}{y^2+x^2},$$

$$\text{N} = \frac{ay}{\sqrt{y^2+x^2}} - \frac{x}{y^2+x^2} + 3by^2,$$

$$\because \int \text{M}dx = a\sqrt{y^2+x^2} - \tan.^{-1}\frac{y}{x} + \text{Y},$$

$$\int \text{N}dy = a\sqrt{y^2+x^2} - \tan.^{-1}\frac{y}{x} + by^3 + \text{X},$$

$\because$ X = 0, Y $= by^3$, and therefore the integral is

$$a\sqrt{y^2+x^2} - \tan.^{-1}\frac{y}{x} + by^3 = \text{C}.$$

Laplace uses this integration in his proof of the principle of the composition of force. See Mec. Cel. liv. i. ch. i.

Ex. 4. $(\sin.y + y\cos.x)dx + (\sin.x + x\cos.y)dy = 0,$

$$\int \text{M}dx = x\sin.y + y\sin.x + \text{Y},$$
$$\int \text{N}dy = y\sin.x + x\sin.y + \text{X},$$

$\because$ X = 0 and Y = 0, and the integral is

$$y\sin.x + x\sin.y = \text{C}.$$

Ex. 5. $(2\text{A}y + \text{B}x + \text{D})dy + (2\text{C}x + \text{B}y + \text{E})dx = 0,$

$$\text{A}y^2 + \text{B}xy + \text{C}x^2 + \text{D}y + \text{E}x + \text{F} = 0.$$

II.

Equations in which the variables are separable.

Ex. 1. $\sqrt{1+y^2} \cdot dx - xdy = 0$. Dividing by $x\sqrt{1+y^2}$,

$$\frac{dx}{x} - \frac{dy}{\sqrt{1+y^2}} = 0,$$

which is immediately integrable *.

Ex. 2. $(\text{A}x + \text{B}y)dy + (\text{A}'x + \text{B}'y)dx = 0.$

Let $\frac{y}{x} = z$, and divide by $\text{A}x + \text{B}y$, $\therefore$

$$dy + \frac{\text{A}' + \text{B}'z}{\text{A} + \text{B}z}dx = 0;$$

but $dy = zdx + xdz$. Hence the equation is reduced to the form

$$\text{x}dx + \text{z}dz.$$

Ex. 3. $ay^m dy + (x^m + by^m)dx = 0$. If $\frac{y}{x} = z$,

$$dy + \frac{1+bz^m}{az^m}dx = 0,$$

$$\therefore \frac{dx}{x} + \frac{az^m dz}{az^{m+1} + bz^m + 1} = 0.$$

Ex. 4. $xdy - ydx = \sqrt{x^2 + y^2} \cdot dx$. Let $\frac{y}{x} = z$, $\therefore$

$$dy - zdx = dx\sqrt{1 + z^2},$$

$$\therefore \frac{dx}{x} = \frac{dz}{\sqrt{1+z^2}}.$$

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* In general, in examples we shall proceed no further than the reduction of the equation to one which is integrable by a former rule.

Ex. 5. Find the curve whose area $= \frac{y^3}{x}$. Hence

$$\int y dx = \frac{y^3}{x}.$$

Differentiating and multiplying by x^2,

$$x^2 y dx = 3xy^2 dy - y^3 dx,$$

$$(x^2 y + y^3)dx - 3xy^2 dy = 0,$$

which being homogeneous, let $y = xz$, ∴

$$\frac{dx}{x} - \frac{3zdz}{1-2z^2} = 0,$$

$$\therefore x^4(1 - 2z^2)^3 = c,$$

$$\therefore (x^2 - 2y^2)^3 = cx^2,$$

which is the equation of the sought curve.

Ex. 6. $(3x + 2y)dx - (2x + y)dy = 0$. Let $y = zx$, ∴

$$\frac{dx}{x} - \frac{(2+z)dz}{3-z^2} = 0,$$

$$\therefore \frac{x^2}{c} = \frac{(\sqrt{3}x + y)^{\frac{1}{\sqrt{3}}+\frac{1}{2}}}{(\sqrt{3}x - y)^{\frac{1}{\sqrt{3}}-\frac{1}{2}}}.$$

III.

Linear equations of the form (314.).

Ex. 1. $dy + (y - ax^3)dx = 0$. In this case, by (314.), $X = 1$, $X' = -ax^3$, ∴

$$\int X dx = x, \therefore \int e^{\int X dx} X' dx = - a\int e^x x^3 dx.$$

But

$$a\int e^x x^3 dx = ae^x(x^3 - 3x^2 + 6x - 6).$$

Hence the sought integral is

$$y = ce^{-x} + a(x^3 - 3x^2 + 6x - 6).$$

Ex. 2. $(1 + x^2)dy - (yx + a)dx = 0$. Hence

$$dy - \frac{yx + a}{1 + x^2}dx = 0.$$

Here we have

$$\text{x} = -\frac{x}{1+x^2}, \quad \text{x}' = -\frac{a}{1+x^2}.$$

Hence $\int \text{x}dx = -\frac{1}{2}l(1+x^2) = -l\sqrt{1+x^2}$. Also,

$$\int e^{\int \text{x}dx}\text{x}'dx = -\int (1+x^2)^{-\frac{1}{2}} \cdot \frac{adx}{1+x^2} = -\int \frac{adx}{(1+x^2)^{\frac{3}{2}}},$$

$$\int \frac{adx}{(1+x^2)^{\frac{3}{2}}} = \frac{ax}{\sqrt{1+x^2}} + \text{c}.$$

Hence the sought integral is

$$y = ax + \text{c}\sqrt{1+x^2}.$$

Ex. 3. $dy - \left(\frac{a}{1-x}y + b\right)dx = 0$. In this case

$$\text{x} = -\frac{a}{1-x} \quad \text{x}' = -b,$$

$$\therefore \int \text{x}dx = al(1-x), \quad \therefore e^{\int \text{x}dx} = (1-x)^a,$$

$$e^{-\int \text{x}dx} = \frac{1}{(1-x)^a}.$$

Hence the sought integral is

$$y = \frac{c}{(1-x)^a} - \frac{b(1-x)}{1+a}.$$

IV.

Cases of RICCATI'S *equation* (315.).

Ex. 1. Let $dy + (y^2 - a^2)dx = 0$,

$$\therefore dx = \frac{dy}{a^2-y^2},$$

$$\therefore e^{-2ax}\left(\frac{y+a}{y-a}\right) = \text{c}.$$

Ex. 2. $dy + (y^2 - a^2x^{-4})dx = 0$. Let

$$y = \frac{1}{x} + \frac{z}{x^2}, \text{ and } x = \frac{1}{t},$$

$$\therefore e^{\frac{2a}{x}} \left\{ \frac{x^2y - x + a}{x^2y - x - a} \right\} = \text{c}.$$

Ex. 3. $dy + (y^2 - a^2x^{-\frac{4}{3}})dx = 0$. In this case the exponent $-\frac{4}{3}$ comes under the character $\frac{-4n}{2n+1}$, since they agree when $n = 1$ (315.). Let

$$x = t^{-3}, \quad y = -3a^2z^{-1},$$

$$\therefore dz + (z^2 - 9a^2t^{-4})dt = 0,$$

$$\therefore e^{-6ax^{\frac{1}{3}}} \left\{ \frac{y(1 + 3ax^{\frac{1}{3}}) + 3a^2x^{-\frac{1}{3}}}{y(1 - 3ax^{\frac{1}{3}}) + 3a^2x^{-\frac{1}{3}}} \right\} = \text{c}.$$

V.

Equations rendered integrable by a multiplier.

Ex. 1. $(1 + a\sqrt{1 + x^2})dx + 2by\sqrt{1 + x^2}dy = 0$. This equation will be found not to come within the *criterion* (284.), since

$$\frac{d\text{M}}{dy} - \frac{d\text{N}}{dx} = -\frac{2byx}{\sqrt{1+x^2}}.$$

But since $\text{N} = 2by\sqrt{1 + x^2}$, $\therefore$

$$\frac{1}{\text{N}}\left(\frac{d\text{M}}{dy} - \frac{d\text{N}}{dx}\right) = -\frac{x}{1+x^2},$$

which being a function of x alone (321.), Case 3°, the equation will become integrable if multiplied by a function of x. To determine this function, let it be u. By (321.), Case 3°,

$$lu = -\int\frac{xdx}{1+x^2} = -\tfrac{1}{2}l(1 + x^2),$$

$$\therefore u = \frac{1}{\sqrt{1+x^2}}.$$

Multiplying by this, we find

$$\left(\frac{1}{\sqrt{1+x^2}}+a\right)dx+2by\,dy=0,$$

which is integrable.

Ex. 2. $x^3dy+(4x^2y-(1-x^2)^{-\frac{1}{2}})dx=0$. In this case

$$\frac{d\text{M}}{dy}-\frac{d\text{N}}{dx}=x^2,$$

$$\therefore\ \frac{1}{\text{N}}\left\{\frac{d\text{M}}{dy}-\frac{d\text{N}}{dx}\right\}=\frac{1}{x}.$$

This being a function of x alone, a factor may be determined, which will render the equation integrable. By (321.), Case 3°, this factor is

$$u=e^{\int\frac{dx}{x}},\ \therefore u=e^{lx},\ \therefore u=x.$$

Hence the equation being multiplied by x, gives

$$x^4dy+[4x^3y-x(1-x^2)^{-\frac{1}{2}}]dx=0,$$

which comes within the criterion, since

$$\frac{d\text{M}}{dy}-\frac{d\text{N}}{dx}=4x^3-4x^3=0.$$

SECTION XX.

Singular solutions.

(324.) Two methods of deducing differential equations from their primitives or integrals have been explained in Section XVI., one by direct differentiation, and the other by eliminating a constant between the primitive equation and its immediate differential. Let $\text{F}(xyc)=0$ be the primitive equation, c being the constant designed for elimination, and let $\text{F}'(xycp)=0$ (where $p=\frac{dy}{dx}$) be its immediate differential, obtained by differentiating the former for x and y.

Eliminating c by these two equations, let the result be $f'(xyp) = 0$. This equation being independent of c, will evidently be the same whatever value be ascribed to c in the primitive equation. When c in the equation $F(xyc)=0$ is taken as an indeterminate or arbitrary constant, this equation is called the *complete integral* of the differential equation $f'(xyp) = 0$; but when a particular value is ascribed to c, it is called a *particular integral*, as being only a case of the equation in its general state.

(325.) It does not, however, necessarily follow that the *complete integral*, including an arbitrary constant, contains *all* the primitive equations from which the differential equation $f'(xyp) = 0$ may be derived. It certainly includes all the *particular integrals*, that is, all those which involve an arbitrary constant; but there may be certain other primitive equations, which, containing no arbitrary constant, are not included under the formula $F(xyc) = 0$, and yet from which the equation $f'(xyp) = 0$ may be deduced. Such equations are therefore entitled to be considered as integrals equally with the equation $F(xyc) = 0$. Such integrals* are called *particular* or *singular solu-*

* There is a species of solutions which may satisfy a differential equation besides those which are considered in this section. Let $Mdx + Ndy = 0$ be a differential equation, and let any function of the variables, as $f(xy)$, be supposed to be a common factor of M and N. It is obvious that $f(xy) = 0$ and $Mdx + Ndy = 0$ will be fulfilled at the same time. In this point of view $f(xy) = 0$ may be considered as a solution of $Mdx + Ndy = 0$. Such solutions, however, are not comprised in the present investigation. They may always be found by determining the common divisors of M and N. These solutions ought not to be termed integrals of the proposed equation, because it does not follow, that

tions, as opposed to the integral $F(xyc) = 0$, including the arbitrary constant, which is called the *general solution*.

(326.) This species of solutions occasioned considerable embarrassment to the earlier analysts, and were held as a kind of analytical paradoxes. Euler considered them as forming exceptions to the general rules of the calculus, and gave methods of distinguishing them from ordinary integrals. Clairaut also determined a class of differential equations which admit of singular solutions *. The complete exposition of the theory of singular solutions, of their connexion with the complete or general solution, and of the circumstances from which they derive their origin, was the work of Lagrange.

(327.) Let $F(xyu) = 0$ be an equation between the variables x, y, u being a function of x and y, and let u be supposed to enter this equation in the same manner as the constant c enters the equation $F(xyc) = 0$. So that taking xy as given quantities, the one is the same function of u as the other is of c.

Also, let u be such a function of the variables x and y, that the equation being differentiated for x and y, the function u shall enter the differential equation $F'(xyup) = 0$ in the same manner as the constant c enters the differential equation $F'(xycp) = 0$; that is, so that if x, y, and p were taken as constant, the one would be the same function of u as the other is of c. The method of determining what function of x and y will satisfy this condition shall be explained presently.

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being differentiated, their differentials would be equivalent to the proposed, which is the specific character of a primitive or integral.

* See Sect. XXII. (350.)

(328.) Since then the two systems of equations

$$F(xyc) = 0, \qquad F'(xycp) = 0,$$
$$F(xyu) = 0, \qquad F'(xyup) = 0,$$

are such that they become identical by changing c into u, or *vice versa*; it follows that if c be eliminated by the first system, and u by the second, the same equation between x, y, and p will be the result.

Let this equation be

$$f'(xyp) = 0.$$

Now it is plain that this is a differential equation of the first order derived equally from the first equations of the two systems, which have therefore equal claims to be considered as its integrals. By the definitions already given, the equation

$$F(xyc) = 0$$

is its *complete integral*, or *general solution*; and such cases of the equation

$$F(xyu) = 0$$

as are not comprised under the former (for it will presently appear that *some* are), are *particular* or *singular* solutions.

(329.) The condition which limits the function u is the identity of the equations

$$F'(xycp) = 0, \quad F'(xyup) = 0.$$

The one may be expressed as the sum of the two partial differentials of $F(xyc)$ taken with respect to the variables x and y successively; the other as the sum of the three partial differentials of $F(xyu)$ taken with respect to x, y, and u successively (95.). Since c and u enter $F(xyc)$ and $F(xyu)$ in exactly the same manner, they must necessarily also enter their partial differential coefficients taken successively with respect to x and y in exactly the same manner. Hence the sum of the partial differentials of $F(xyc)$ taken successively with respect to x and y must be the same function of c as the sum of the partial differentials of $F(xyu)$ with respect to

x and y is of u. In order, therefore, that the two equations should be identical, it is necessary that the partial differential of $\mathrm{F}(xyu)$, taken with respect to u, should $= 0$. Let the partial differential coefficient of $\mathrm{F}(xyu)$ taken with respect to u be U; the partial differential is $\mathrm{U}du$. Now, since u is by hypothesis not a constant, du is not $= 0$; therefore the condition $\mathrm{U} = 0$ must be fulfilled. An example will make these principles easily apprehended. Let

$$\mathrm{F}(xyc) = x^2 + y^2 - 2cy - c^2 = 0,$$
$$\therefore \mathrm{F}'(xycp) = (y - c)p + x = 0.$$

Eliminating c, we obtain

$$f'(xyp) = (x^2 - 2y^2)p^2 - 4xyp - x^2 = 0.$$

Now let c be changed into u, and we have

$$\mathrm{F}(xyu) = x^2 + y^2 - 2yu - u^2 = 0,$$
$$\mathrm{F}'(xyup) = xdx + (y - u)pdx - (y + u)du = 0,$$

observing that $dy = pdx$.

The functions $\mathrm{F}'(xycp)$ and $\mathrm{F}'(xyup)$ will be rendered identical by the condition $u = -y$, in which case, the elimination of u by the latter equations, and that of c by the former, will both lead to the same differential equation,

$$f'(xyp) = (x^2 - 2y^2)p^2 - 4xyp - x^2 = 0.$$

The condition $u = -y$ changes $\mathrm{F}(xyu)$ into

$$x^2 + 2y^2 = 0,$$

which is therefore a singular solution of the differential equation.

From the preceding observations, we may therefore infer generally, that if the *general solution*, cleared of radicals,

$$\mathrm{F}(xyc) = 0,$$

of any proposed differential equation,

$$f'(xyp) = 0,$$

be differentiated, considering the arbitrary constant c alone variable, and that the partial differential

$$cdc = 0$$

be thus determined; the values of c, which satisfy this condition, being substituted for c in the general solution

$$\text{F}(xyc) = 0,$$

will give equations, amongst which all *singular solutions* will be found.

(330.) It does not, however, necessarily follow that all the equations resulting from such substitutions are singular solutions. Such equations may be cases of the general solution in which the arbitrary constant receives a particular value, in which case they are particular integrals, and not singular solutions.

The partial differential coefficient C is in general composed of c, the variables x, y, and the constants of the proposed differential equation. It may, however, happen in particular cases, that C does not contain the variables x, y. In such cases the values of c derived from $\text{C} = 0$ are functions of constant quantities, and are therefore themselves constant. The substitution of such values for c in the general solution would, therefore, only give particular integrals, and not singular solutions.

Again, if C contained the variables, or either of them, and the elimination of one of them between the equation $\text{C} = 0$ and the general solution $\text{F}(xyc) = 0$, were to give a result independent of the other variable, this would determine a particular constant value for c, which, substituted in the general solution, would give a particular integral.

Further, the partial differential coefficient C may be independent of c, which will always happen when c enters the general solution $\text{F}(xyc) = 0$ in the first degree. In this case the general solution must have the form

$$\text{Q} + \text{C}c = 0,$$

$$\therefore c = -\frac{\text{Q}}{\text{C}}.$$

If C be not a factor of Q, the condition $\text{C} = 0$ renders c in-

finite, and therefore gives the particular case of the general solution, in which the arbitrary constant is infinite. The result is, in this case, a particular integral.

But if C be a factor of Q, then the condition C = 0 is itself a solution of the proposed differential equation, which being

$$C d\text{Q} - \text{Q} dC = 0,$$

is obviously satisfied by C = 0. To determine in this case whether C = 0 be a particular integral or a singular solution, let one of the variables be eliminated by the equation C = 0 and the general solution, and the value of c be determined by the resulting equation. The equation C = 0 is a singular solution if this be variable, and otherwise not.

(331.) The principles thus established furnish us with the solution of the problem, "Given the general solution [$\text{F}(xyc) = 0$] of a differential equation [$f'(xyp) = 0$] to find its singular solutions, if any such exist." Clear the general solution of radicals, and take its partial differential with respect to the arbitrary constant considered as a variable; eliminate the arbitrary constant from the general solution by the *variable* values of it, which satisfy the general solution, and the partial differential equation before mentioned; the equations resulting from such elimination are *singular solutions*.

(332.) It is obvious that the condition C = 0 is that by which the general solution $\text{F}(xyc) = 0$ solved for c as an unknown quantity will have equal roots. (Geometry, Art. 580.). If, therefore, by means of the singular solution and the general one, either of the variables be eliminated, the result will have equal factors. Thus, in the example given in (329.), if x be eliminated, the result will be

$$y^2 + 2cy + c^2 = (y + c)^2 = 0.$$

Also, since the equality of the roots is produced by the dis-

appearance of radicals, it follows, that if the general solution be solved for the arbitrary constant, and the radicals which enter the values be assumed = 0, the resulting equations will be singular solutions, provided they satisfy the proposed equation, and are not cases of the general solution.

It may here be observed generally, that tests for determining any equation to be a singular solution are twofold:

1°. That it satisfy the proposed differential. That is, that its differential shall be identical with the proposed. This is necessary, in order that it may be a solution *at all.*

2°. That it be not a case of the general solution, in which case it would be a particular integral, and not a singular solution.

(333.) If the partial differentials of the primitive equation $\text{F}(xyc) = 0$ be taken with respect to the variables successively, c being considered as a variable function of x and y, we obtain two equations of the forms

$$\frac{d\text{v}}{dx} + \frac{d\text{v}}{dc}\frac{dc}{dx} = 0,$$

$$\frac{d\text{v}}{dy} + \frac{d\text{v}dc}{dc\,dy} = 0,$$

where v represents the function $\text{F}(xyc)$. If $\text{F}(xyc) = 0$ be a singular solution, it has been already proved (330.), that

$$\frac{d\text{v}}{dc} = 0.$$

Hence we find

$$\frac{dc}{dx} = \frac{d\text{v}}{dx} \div \frac{d\text{v}}{dc} = \infty,$$

$$\frac{dc}{dy} = \frac{d\text{v}}{dy} \div \frac{d\text{v}}{dc} = \infty.$$

This furnishes another character for the determination of singular solutions. Let the general solution be differentiated, and the partial differential coefficients obtained by considering the arbitrary constant successively as a function of each

variable. The conditions which render these coefficients infinite will give two equations

$$\frac{dc}{dx} = \infty, \quad \frac{dc}{dy} = \infty;$$

from which, by eliminating c, equations may be obtained, which will, in general, be singular solutions. They must, however, be submitted to the tests in (332.).

This presents again the property by which singular solutions arise from the disappearance of radicals in the values of c; for when radicals enter the value of c, they will always appear in the denominators of the values of $\frac{dc}{dx}$ and $\frac{dc}{dy}$.

(334.) Singular solutions have the peculiar property of rendering infinite those multipliers which render the differential equation integrable. This property might lead us to conclude that the investigation of the factors which render an equation integrable would also involve the determination of singular solutions. But this would require the converse of the principle *scil.* that equations which render the multipliers infinite are singular solutions, which is not generally true *.

Let the differential equation

$$\text{M}dx + \text{N}dy = 0$$

be one which admits a singular solution, and let its general solution be $\text{F}(xy) = c$, and its singular solution $\text{F}(xy) = u$. Let z be the factor which renders the equation integrable, ∴

$$z\text{M}dx + z\text{N}dy$$

is the exact differential of $\text{F}(xy)$. If a value of y be derived from $\text{F}(xy) = u$, and substituted in the general solution, c will not continue constant, for if it did, the equation

* Laplace, *Mems. de l'Acad. des Sciences*, 1772.

$F(xy) = u$ would be included in $F(xy) = c$, and therefore would not be a singular solution, which is contrary to hypothesis. Since, therefore, c is not constant, dc is not $= 0$, $\therefore z(Mdx + Ndy)$ is not $= 0$; but by hypothesis $Mdx + Ndy = 0$, $\therefore z$ is infinite, and, therefore, all factors which render the equation integrable are infinite.

(335.) POISSON has demonstrated * in the Journal of the *Polytechnic School*, that by certain transformations, every differential equation of the first order may be rendered divisible by its singular solution; and *vice versa*, that any given singular solution may always be introduced. This subject, however, has not been reduced to a sufficiently simple state to admit of being properly introduced into a treatise so elementary as the present. It is besides of little use in the applications of this science.

(336.) Whenever the genera. olution of a differential equation has been obtained, or is given, the singular solutions may be always derived by the principles which have been just established. It is, however, frequently necessary to be able to pronounce whether a proposed solution be a singular solution, or particular integral, when the general solution is not known, and when therefore the question cannot be decided by an immediate reference to it.

Let $F(yx) = 0$ be a solution of the equation $f'(xyp) = 0$, it is required to determine whether it is a particular integral or a singular solution, the general solution being unknown.

Let the value of y derived from the proposed solution be X, and let the value derived from the general solution be V, the former being a known function of x, and the latter un-

* See also an art. by LEGENDRE, *Memoires de l'Acad. des Sciences*, 1790.

known. The value v includes an arbitrary constant c. Now, if the proposed solution be singular, no value whatever of c can render v and x identical; but if it be a particular integral, there is a certain value c' of c, which will give v = x; so that v − x and $c - c'$ will vanish together. Hence it follows that

$$\mathrm{v} - \mathrm{x} = \mathrm{A}(c - c')^m,$$

where A is a quantity which becomes neither infinite nor $= 0$, when $c - c' = 0$ and $m > 0$. Let $(c - c')^m = h$, ∴

$$\mathrm{v} - \mathrm{x} = \mathrm{v}'h + \mathrm{v}''h^{\mu} + \cdots$$

$$\therefore \mathrm{v} = \mathrm{x} + \mathrm{v}'h + \mathrm{v}''h^{\mu} + \cdots$$

This may be considered as the development of y in the general solution.

Let the proposed differential equation resolved for dy be $dy = pdx$. This equation ought to be satisfied by the general solution independently of h. Let the value of y found from this solution be $\mathrm{x} + k$, and p being expressed in powers of k, we have

$$p = \mathrm{P} + \mathrm{P}'k^m + \mathrm{P}''k^n + \cdots$$

where the exponents are ascending and positive. For p is not infinite when $k = 0$, since the equation $y = \mathrm{x}$ (which does not render $p = \infty$) renders the equation $dy = pdx$ identical, and ∴ $d\mathrm{x} = \mathrm{P}dx$.

When $y = \mathrm{x} + k$, ∴

$$d\mathrm{x} + dk = (\mathrm{P} + \mathrm{P}'k^m + \mathrm{P}''k^n + \cdots)dx,$$

$$d\mathrm{x} = \mathrm{P}dx,$$

$$\therefore dk = (\mathrm{P}'k^m + \mathrm{P}''k^n + \cdots)dx.$$

Substituting for k its value,

$$\mathrm{v} - \mathrm{x} = \mathrm{v}'h + \mathrm{v}''h^{\mu} + \cdots$$

we obtain

$$hd\mathrm{v}' + h^{\mu}d\mathrm{v}'' + \cdots = \left\{\begin{array}{l} \mathrm{P}'h^m(\mathrm{v}' + \mathrm{v}''h^{\mu-1} + \cdots)^m dx \\ + \mathrm{P}''h^n(\mathrm{v}' + \mathrm{v}''h^{\mu-1} + \cdots)^n dx \\ + \cdots\cdots\cdots \end{array}\right\}$$

This equation must be satisfied independently of h when $y = \text{x}$ is a particular integral. If this be not possible, it is a singular solution.

Equating the terms with the lowest exponents, we obtain

$$d\text{v}' = \text{P}'\text{v}'^{m}h^{m-1}dx,$$

which is only independent of h when $m - 1 = 0$, $\therefore m = 1$. In this case

$$d\text{v}' = \text{P}'\text{v}'dx,$$
$$\therefore l\text{v}' = \int \text{P}'dx,$$
$$\text{v}' = e^{\int \text{P}'dx}.$$

If $m - 1 > 0$, the terms cannot be identified; but $hd\text{v}'$ may disappear by supposing $d\text{v}' = 0$, $\therefore \text{v}'$ constant. Then, if $\mu = m$, $\therefore d\text{v}'' = \text{P}'dx$, $\therefore \text{v}'' = \int \text{P}'dx$, and, in a similar way, the other terms may be found.

Thus it appears, that if $m - 1$ be not < 0, the two series may be identified, and therefore the proposed solution is a particular integral. But this cannot take place if $m - 1 < 0$, that is, if m be a proper fraction; since, in that case, the term $\text{P}'\text{v}'^{m}h^{m}dx$ cannot be identified with $hd\text{v}'$, or any of the following terms. In this case, therefore, the proposed solution is singular.

(337.) This investigation furnishes a new criterion for the detection of singular solutions, and one which is altogether independent of the knowledge of the general solution. It appears from what has been proved, that, if upon changing y into $y + k$ in the proposed differential equation $f'(xyp) = 0$ solved for p, and developing the corresponding value of p in powers of k, the first exponent of k be less than unity, the equation between the variables which fulfils this condition will be a singular solution, provided it satisfy the proposed equation. By (55.), it follows, that the condition on which the first exponent in the development is less than unity is

$$\frac{dp}{dy} = \infty.$$

And by similar reasoning applied to the other variable, it follows, that singular solutions may be determined by the condition

$$\frac{dp}{dx} = \infty.$$

Thus, if $\frac{dp}{dy}$ or $\frac{dp}{dx} = \frac{\text{M}}{\text{N}}$, every singular solution must render $\text{N} = 0$, and must therefore be a divisor of it. Also, every factor of N, which is not also a factor of M, and which satisfies the proposed equation, is a singular solution.

The solution of the proposed differential equation for p may be avoided by differentiating it for x, y, and p.

Let $f(xyp) = \text{U} = 0$, $\because$

$$\frac{d\text{U}}{dx}dx + \frac{d\text{U}}{dy}dy + \frac{d\text{U}}{dp}dp = 0,$$

$$\therefore \frac{dp}{dy} = -\frac{\frac{d\text{U}}{dy}}{\frac{d\text{U}}{dp}},$$

$$\frac{dp}{dx} = -\frac{\frac{d\text{U}}{dx}}{\frac{d\text{U}}{dp}}.$$

If the equation $f(xyp) = 0$ have been previously cleared of radicals, the condition under which these coefficients will become infinite is

$$\frac{d\text{U}}{dp} = 0.$$

But otherwise, the condition may also be satisfied by the equations

$$\frac{1}{\frac{d\text{U}}{dx}} = 0, \quad \frac{1}{\frac{d\text{U}}{dy}} = 0 \;.\;.\;.\;.\; [a].$$

The elimination of p by the proposed differential equation $f(xyp) = 0$, and any one of the three preceding equations, will determine a singular solution, provided that the result satisfy the proposed equation.

(338.) The former of the equations [α] should be used when the proposed differential equation does not contain y, and the latter when it does not contain x. The one determines singular solutions of the form $x = c$, and the other of the form $y = c$.

(339.) From what has been already observed on the method of deducing singular solutions from general ones (333.), it is obvious that the conditions

$$\frac{dp}{dx} = \infty, \quad \frac{dp}{dy} = \infty,$$

will always be satisfied by making the radicals which enter the values of p derived from the proposed equation $= 0$.

In applying these conditions, the equation should be previously solved for p, otherwise it will be necessary to eliminate p between either of these and the proposed equation.

If the equation

$$\frac{d\upsilon}{dx}dx + \frac{d\upsilon}{dy}dy + \frac{d\upsilon}{dp}dp = 0$$

be solved for dp, we obtain

$$dp = -\frac{\frac{d\upsilon}{dx}dx + \frac{d\upsilon}{dy}dy}{\frac{d\upsilon}{dp}},$$

$$\therefore \frac{dp}{dx} = \frac{d^2y}{dx^2} = -\frac{\frac{d\upsilon}{dx}dx + \frac{d\upsilon}{dy}dy}{\frac{d\upsilon}{dp}dx}.$$

The conditions already established for the determination of singular solutions,

$$\frac{dv}{dx}dx + \frac{dv}{dy}dy + \frac{dv}{dp}dp = 0,$$

$$\frac{dv}{dp} = 0,$$

render both numerator and denominator of the former expression $= 0$. Hence, *singular solutions* must always render the second differential coefficient $\frac{0}{0}$.

If the differential coefficient p enter the proposed equation (having been previously cleared of radicals) only in the first power, it is impossible that the equation can admit a singular solution. For p will not, in that case, enter $\frac{dv}{dp} = 0$, which will not, therefore, be sufficient to eliminate p from the proposed equation, on which elimination the determination of a singular solution depends. The same remark extends to equations which are linear with respect to p, but which involve radicals; and it may in general be concluded, that no linear equation, properly so called of any order, allows of singular solutions.

(340.) The connexion of the singular solution with the general one has been determined by considering the arbitrary constant in the general solution as a variable. Taking the general solution as the equation of a curve, the character, magnitude, and position of which will depend upon the values of the constants, and among others, of the arbitrary constant, if a succession of values be ascribed to it, the general solution will represent a succession of curves corresponding to these values; and the equation may be considered as applying to the consecutive intersections of these curves. If, then, the condition of continuity be introduced, and the arbitrary constant be considered as variable, the equation will represent a curve which will include or exclude all the others, and touch them. The general solution,

when any particular value is ascribed to the constant, represents one of the former curves, and determines a relation between the variables, which is expressed by the co-ordinates of any point upon it. But in the other case, the constant is replaced by a variable function of the co-ordinates of the point of contact. The tangent at the point of contact is the same for both curves, being determined by the value of the differential coefficient p, which preserves the same value, whether the arbitrary constant be considered as variable or not in the primitive equation; whence it follows, that by eliminating the constant between the primitive equation and its differential with respect to the constant, the resulting equation between the variables, which is the singular solution, represents the line of contact of the curves comprised in the general solution.

(341.) In general, then, from the results of this section it follows:

1°. That two conditions are indispensable, in order that any equation between the variables may be a *singular solution* of a given differential equation; 1st, That it *be* a solution, that is, that it *satisfy* the proposed differential equation, for otherwise, it is not *a solution* at all; and 2dly, That it be not contained in the general solution, for if it be, it is a *particular integral*, and not a general solution (332.).

2°. That if the general solution be differentiated with respect to the arbitrary constant and its differential coefficient equated with o, and by this equation and the general solution the arbitrary constant be eliminated, *singular solutions* may be found among the factors of the resulting equation (331.).

3°. If the general solution be solved for the arbitrary constant, so that this constant may be expressed as a function of the two variables, and that its two partial differential coefficients taken with respect to each variable be found,

singular solutions may be found from the equations which render either or both of these infinite (333.).

4°. The condition under which the factor, which renders the equation integrable, becomes infinite, *may* contain singular solutions (334.).

5°. If a differential equation be differentiated with respect to the differential coefficient, and this coefficient eliminated by the equation thus obtained, and the differential equation itself, the resulting equation between the variables may contain singular solutions (337.).

6°. If a differential equation be differentiated with respect to either of the variables, and by the equation which renders the partial differential coefficient thus found infinite, and the proposed differential equation, the differential coefficient be eliminated, the resulting equation between the variables *may* contain singular solutions (337.).

7°. If a differential equation be algebraic, and include irrational functions, singular solutions *may* be found amongst the equations which make these radicals disappear. This may be effected by the suffixes or coefficients of the radicals vanishing (339.).

8°. The conditions which render the second differential coefficient $\frac{0}{0}$ may contain singular solutions (339.).

(In the last seven observations we have expressed ourselves in a *contingent* sense, since the results must severally fulfil the two conditions of 1°, in order to be singular solutions, which in some cases they do not.)

9°. It is of as much importance to determine the singular solutions as the *general solution*, since, in many cases, the true solution of the proposed problem is to be found amongst them, and not in the general solution (340.). See Section XXII, Ex. 11. to Ex. 15.

10°. Geometrical problems, the object of which is the determination of curves touching any number of curves of the same kind, but differing from each other by the parameter, or some other constant part, are solved by singular solutions (340.).

SECTION XXI.

Of the integration of differential equations of the first order, and which exceed the first degree.

(342.) It appears by the ordinary process of differentiation, that no differential equation, obtained *directly* by differentiating the primitive equation, can exceed the first degree. But when between the primitive equation and its immediate differential a constant is eliminated, which enters these equations in any degree superior to the first, the result will be a differential equation of the same order as before, but of a superior degree.

(343.) Every differential equation of the first order, whatever its degree may be, must be comprised in the formula

$$\left(\frac{dy}{dx}\right)^n + B\left(\frac{dy}{dx}\right)^{n-1} + C\left(\frac{dy}{dx}\right)^{n-2} \cdots\cdots M\frac{dy}{dx} + N = 0.$$

Let the roots of this equation be p, p', p'' . . . . Hence it may be expressed

$$\left(\frac{dy}{dx} - p\right)\left(\frac{dy}{dx} - p'\right)\left(\frac{dy}{dx} - p''\right) \cdots\cdots = 0.$$

This equation is resolved into the several equations

$$dy - pdx = 0,$$
$$dy - p'dx = 0,$$
$$dy - p''dx = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Let each of these be separately integrated, and the integrals be $\text{v} = 0$, $\text{v}' = 0$, $\text{v}'' = 0 \cdots\cdot$ Any one of these integrals, or any number of them combined by multiplication, will satisfy the proposed differential equation. For

$$d\text{v} = dy - pdx = 0; \; d(\text{v}\text{v}') = \text{v}d\text{v}' + \text{v}'d\text{v} = 0,$$
$$\because d(\text{v}\text{v}') = \text{v}(dy - p'dx) + \text{v}'(dy - pdx) = 0.$$

It is obvious that these conditions are satisfied, and that the same will apply to the product of any number of them.

(344.) But a difficulty presents itself from the consideration that an arbitrary constant is introduced in each integration, and that therefore n arbitrary constants are introduced in the integration of a differential equation of the first order, which seems contrary to the principles in the general theory of differential equations. This, however, is accounted for thus: The constant, by the elimination of which the differential equation of the nth order was obtained, must have entered the primitive equation in the nth degree, and therefore it had n different values derivable from that equation; the n arbitrary constants, therefore, thus introduced, are only these n different values of the constant eliminated.

(345.) The n differential equations of the first degree, into which the proposed equation has been resolved, may also be accounted for by mere differentiation. Let the primitive equation be imagined to be solved for the constant, of which, therefore, n values will be obtained. Upon differentiating the equation, each of these values will give a distinct equation of the first order and first degree. These

equations are no other than the simple factors of the differential equation of the nth degree.

(346.) By what has been just explained, it appears that the integration of differential equations of the first order and superior degrees depends on the resolution of algebraic equations. But as our powers in that department of analysis are extremely limited, several artifices have been suggested to elude the necessity of the resolution of the differential equation; we shall therefore explain the principal of these.

(347.) If the differential equation only contain one of the variables x, and the differential coefficient p, and can be resolved for x, it will give

$$x = \text{F}(p).$$

Now, since $dy = pdx$, integrating by parts, we find $y = px - \int xdp = px - \int \text{F}(p)dp$. Thus, the integration of the equation is reduced to that of the formula $\text{F}(p)dp$, which can be effected by the rules already established.

(348.) If the proposed differential equation contain both variables, one y, entering it only in the first degree, then solving the equation for y, we find

$$y = \text{F}(xp) = \text{V},$$

$$\therefore dy = \frac{d\text{V}}{dx}dx + \frac{d\text{V}}{dp}dp.$$

But $dy = pdx$, $\therefore$

$$\left(\frac{d\text{V}}{dx} - p\right)dx + \frac{d\text{V}}{dp}dp = 0.$$

If this equation can be integrated, an equation of the form

$$f(xp) = 0$$

will be obtained.

By this and the proposed equation, p being eliminated, an equation between x and y will be the result, which is the sought integral.

SECTION XXII.

Praxis on singular solutions, and the integration of differential equations of the first order and superior degrees.

(349.) Ex. 1. Let the proposed equation be

$$p^2y + 2px - y = 0,$$

where $p = \frac{dy}{dx}$. Hence

$$p = \frac{-x \pm \sqrt{x^2+y^2}}{y},$$

$$\therefore \frac{ydy+xdx}{\sqrt{x^2+y^2}} = \pm\, dx,$$

which being integrable, gives

$$\pm \sqrt{x^2 + y^2} = x + c,$$

$$\therefore y^2 - 2cx - c^2 = 0 \quad (1)$$

is the general solution.

To determine the singular solutions, let the last equation be differentiated for c (331.). This gives

$$c + x = 0,$$

which, by eliminating c by the general solution, becomes

$$y^2 + x^2 = 0.$$

The value of c being variable, and this last solution not being a case of the general one, it is a *singular solution*.

The same result might be obtained by (332.) solving the equation (1) for c, and making the radical $= 0$; thus,

$$c = - \quad x \sqrt{x^2 + y^2},$$

$$\therefore x^2 + y^2 = 0.$$

If we examine the general solution (1) by the tests established in (333.), we find

$$\frac{dc}{dx} = -\frac{c}{c+x} = \infty,$$

$$\frac{dc}{dy} = \frac{y}{c+x} = \infty,$$

which give the singular solution already determined.

The singular solution may be obtained immediately from the proposed equation without the general solution by the method explained in (337.). By differentiating for p and x, and p and y, we find

$$\frac{dp}{dx} = -\frac{p}{py+x} = \infty,$$

$$\frac{dp}{dy} = \frac{1-p^2}{py+x} = \infty;$$

these conditions are both fulfilled by

$$py + x = 0.$$

Eliminating p by this and the proposed differential equation, we obtain

$$x^2 + y^2 = 0,$$

the singular solution.

The same result may be still more readily found (337.) by differentiating the proposed equation for p only, which gives

$$py + x = 0,$$

from which we obtain the singular solution as before.

Ex. 2. Let the general solution of a differential equation be

$$y^2 + x^2 = 2cx,$$

it is required to assign the singular solution.

Differentiating for c, we find $x = 0$. Since this is independent of c, it only gives the particular integral corresponding to an infinite value of c (330.).

Ex. 3. Let the general solution be

$$(x^2 + y^2 - b)(y^2 - 2cy) + (x^2 - b)c^2 = 0.$$

Differentiating for c, we find

$$c = \frac{y(x^2+y^2-b)}{x^2-b},$$

$$\therefore y^2(y^2 + x^2 - b) = 0.$$

This being only the case of the general solution in which $c = 0$, it is a particular integral.

Ex. 4. $(1 + p^2)x = 1$. Hence

$$x = \frac{1}{1+p^2} = \text{F}(p).$$

Hence by the formula (347.),

$$y = px - \int \text{F}(p)dp,$$

$$\therefore y = px - \tan.^{-1} p + c.$$

Eliminating p by this and the given equation

$$y = \sqrt{x - x^2} - \tan.^{-1}\frac{\sqrt{1-x}}{\sqrt{x}} + c.$$

Ex. 5. Given a general solution

$$y = x + (c-1)^2\sqrt{x},$$

to determine the singular solution. Let it be differentiated for c. Hence

$$2(c-1)\sqrt{x} = 0,$$

$$\therefore c = 1.$$

Hence $y = x$. But since this is contained in the general solution, it is only a particular integral, and not a singular solution.

Ex. 6. To determine the singular solution of

$$(x^2 - b)p^2 - 2xyp = x^2,$$

of which the general solution is

$$x^2 - 2cy - b - c^2 = 0.$$

Differentiating for c, we obtain $c = -y$. Hence

$$x^2 + y^2 = b.$$

Since this satisfies the proposed differential equation, and is not included in the general solution, it is a singular solution.

The same might be also immediately obtained from the proposed equation by solving it for p, and making the quantity under the radical $= 0$ (339.).

Ex. 7. Let the proposed equation be

$$ydx - xdy = x\sqrt{dx^2 + dy^2}.$$

This equation is homogeneous with respect to x and y. Let $y = ux$, $\because$

$$u = p + \sqrt{1 + p^2},$$

$$\frac{dx}{x} = \frac{du}{p-u}, \ \because lx = -\,l(p - u) + \int\frac{dp}{p-u},$$

$$\because lx = -\,l\sqrt{1 + p^2} - l(p + \sqrt{1 + p^2}) + lc,$$

$$\because x = \frac{c}{\sqrt{1+p^2}}(p - \sqrt{1 + p^2}),$$

$$\because y = \frac{c}{\sqrt{1+p^2}}\cancel{(p - \sqrt{1 + p^2})}.$$

Eliminating p, we find

$$x = 0, \quad x^2 + y^2 + 2cx = 0.$$

The former is contained in the latter being what it becomes when $c = \infty$. There is in this case no singular solution.

Ex. 8. Let the general solution be

$$c^2 - (x + y)c - c + x + y = 0.$$

Differentiating for c, we find

$$2c - x - y - 1 = 0, \quad \because c = \tfrac{1}{2}(x + y + 1).$$

Hence we find, by eliminating $x + y$,

$$c = 1,$$

$$\because x + y = 0,$$

which is only a particular integral.

Ex. 9. Let the general solution be

$$y = x + (c - 1)^2(c - x)^2.$$

Differentiating for c, we find

$$(c - 1)(c - x)(2c - x - 1) = 0.$$

This is satisfied in three ways,

$$c = 1, \quad c = x, \quad c = \tfrac{1}{2}(x + 1).$$

The first evidently gives a particular integral. The second, although c is variable, gives $y = x$ also a particular case of the general solution. The last, however, gives a singular solution.

Ex. 10. Let the proposed equation be

$$y^2dx^6 - yxdxdy^5 - 2yxdydx^5 + x^2dy^6 + x^2dy^2dx^4 = 0.$$

This being homogeneous, let $y = ux$; which being substituted for y, and the result divided by x^2, gives

$$u^2 - u(p^5 + 2p) + p^2(1 + p^4) = 0,$$

which being solved for u, gives

$$u = p, \text{ or } u = p(1 + p^4).$$

From the latter, we find

$$x = \frac{ce^{\frac{1}{4p^4}}}{p^5}, \quad y = \frac{(1 + p^4)ce^{\frac{1}{4p^4}}}{p^4}.$$

The former gives $y = x$, which is a singular solution.

(350.) The equation $y = px + \text{P}$, P being a function of the differential coefficient was first proposed and integrated by *Clairaut*. Let this equation be differentiated, and we find

$$dy = pdx + xdp + d\text{P}.$$

But $dy = pdx$, $\therefore$

$$xdp + d\text{P} = 0,$$

$$\therefore \left(x + \frac{d\text{P}}{dp}\right)dp = 0.$$

Hence we have $dp = 0$, $\therefore p =$ constant, or

$$x + \frac{d\text{P}}{dp} = 0.$$

By eliminating p by $p = c$, and the given equation, we find

$$y = cx + \text{C},$$

C being the same function of c as P is of p. This is the general solution. Eliminating p by the equation

$x + \frac{dP}{dp} = 0$, and the given equation, we find the singular solution. We subjoin some applications of this formula.

PROP. XCIX.

Ex. 11. *To find the curve, such, that perpendiculars from two given points to the tangent shall contain a rectangle of a given magnitude.*

Let the equation of the tangent be

$$(y' - y) - p(x' - x) = 0,$$

yx being the point of contact. Let the points from which the perpendiculars are drawn be taken upon the axis of x at equal distances $+ c$, and $- c$ on different sides of the origin. Hence the two perpendiculars are

$$-\frac{y+p(c-x)}{\sqrt{1+p^2}},$$

$$-\frac{y-p(c+x)}{\sqrt{1+p^2}}.$$

Let the product of these be b^2, $\therefore$

$$y^2 - p^2(c^2 - x^2) - 2pxy = b^2(1 + p^2).$$

This solved for y, gives

$$y = px \pm \sqrt{b^2 + a^2p^2},$$

where $a^2 = b^2 + c^2$. Hence, if k be an arbitrary constant, the general solution is

$$y = kx \pm \sqrt{b^2 + a^2k^2},$$

which is the equation of the tangents to the sought curves.

The particular solution is

$$a^2y^2 + b^2x^2 = a^2b^2,$$

which is the equation of an *ellipse*.

If the two perpendiculars be supposed to be drawn to

opposite sides of the axis of x, their product is negative, $\therefore b^2 < 0$. In this case the particular solution becomes

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

which is the equation of the hyperbola. This is a well known property of those curves. Geometry (215.).

PROP. C.

Ex. 12. *To find the curve such, that a perpendicular from a given point upon its tangent shall have a constant length.*

The equation of the tangent being represented as before, let the given point be the origin, and let the constant length of the tangent be r, $\therefore$

$$\frac{y-px}{\sqrt{1+p^2}} = r,$$

$$\therefore y = px + r\sqrt{1+p^2}.$$

Hence the singular solution is

$$y^2 + x^2 = r^2.$$

The circle is therefore *unique* in this property.

PROP. CI.

Ex. 13. *To find the curve such, that perpendiculars to a given right line from two given points upon that line drawn to meet the tangent shall include a given rectangle.*

The equation of the tangent being represented as before, let the given line be taken as the axis of x, the origin being at the middle point of the intercept between the given points. Let the distances of the given points from the origin be $+a$ and $-a$. Hence the two perpendiculars

$$y + p(a - x),$$
$$y - p(a + x).$$

Let the constant value of the rectangle under these be b^2. Hence

$$y^2 - p^2(a^2 - x^2) - 2pyx = b^2,$$
$$\therefore y = px \pm \sqrt{b^2 + a^2p^2}.$$

Hence the general solution is

$$y = kx \pm \sqrt{b^2 + k^2a^2},$$

k being an arbitrary constant. And the particular solution is

$$a^2y^2 + b^2x^2 = a^2b^2,$$

which is an ellipse or hyperbola according as b^2 is > 0 or < 0, that is, according as the perpendiculars are at the same or different sides of the axis of x. Geometry (192.).

PROP. CII.

Ex. 14. *To find the curve such, that the locus of the point where a perpendicular from a given point meets its tangent is a circle.*

Let the line through the centre of the supposed circle and the given point be the axis of x, and let the origin be taken at the centre of the circle, the distance of the given point from the centre being c, and the radius of the circle a. Let the angle under the perpendicular, and the axis of x be φ, and the perpendicular z, $\because$

$$a^2 = z^2 + c^2 + 2zc \cos. \varphi.$$

But $\tan. \varphi = -\dfrac{1}{p}$, $\because \cos. \varphi = \dfrac{p}{\sqrt{1+p^2}}$, and

$$z = \frac{y - p(c + x)}{\sqrt{1+p^2}},$$

$$\because a^2 = \frac{[y-p(c-x)]^2}{1+p^2} + c^2 + \frac{2pc[y-p(c-x)]}{1+p^2}.$$

Hence we find

$$[y - p(c - x)]^2 + 2pc[y - p(c - x)] = b^2(1 + p^2),$$

where $a^2 - c^2 = b^2$. Hence

$$y - p(c + x) = - pc \pm \sqrt{a^2p^2 + b^2},$$

$$\therefore y = px \pm \sqrt{a^2p^2 + b^2},$$

the singular solution is therefore

$$a^2y^2 + b^2x^2 = a^2b^2,$$

which is an ellipse or hyperbola, according as $b^2 > 0$ or < 0, that is as $a > c$ or $a < c$. Geom. (223.).

PROP. CIII.

Ex. 15. *To find a curve such, that the locus of the intersection of a perpendicular from a given point with its tangent shall be a right line.*

Let a perpendicular through the given point be drawn to the right line which is the supposed locus, and let these be assumed as axes of co-ordinates, the distance of the point from the supposed locus being a. The equation of the perpendicular to the tangent through the given point is

$$y + \frac{1}{p}(x - a) = 0.$$

The value of y corresponding to $x = 0$ is therefore $\frac{a}{p}$. Hence the intercept of the perpendicular between the given point and supposed locus is

$$\sqrt{a^2 + \frac{a^2}{p^2}} = a\frac{\sqrt{1+p^2}}{p}.$$

Hence

$$\frac{y+p(a-x)}{\sqrt{1+p^2}} = a\frac{\sqrt{1+p^2}}{p},$$

y and x being the co-ordinates of the point of contact. The singular solution of which is

$$y^2 = 4ax,$$

the equation of a parabola.

SECTION XXIII.

Of the integration of differential equations of the second and higher orders.

(351.) One of the circumstances which give facility to the integration of differential equations of the first order, is, that it is immaterial which of the variables is considered as a function of the other, or which is the independent variable. This is not the case when we ascend to the higher orders of differentials (38.), where a transformation is necessary to change the independent variable. As the orders of differential equations rise, the difficulty of their integration increases. It has been proved that every differential equation of two variables *has* an integral; but the discovery of that integral in finite terms when the order of the differential equation exceeds the first, is, in almost every case, attended with considerable difficulty, and, in by far the greater number of cases, has totally baffled the skill of the greatest analysts. It would be impossible in the present state of the science, therefore, to reduce the subject of the present section to a systematical exposition of the integration of differential equations of the higher orders. All that could be done, even in a treatise less elementary than the present, would be, to explain the methods of integrating particular classes of equations which have been discovered by *Euler*, *Lagrange*, *D'Alembert*, and others. To enter at large into the details of these methods would, however, be quite unsuitable to the objects of this work. We shall therefore confine ourselves to the investigation of the methods of integrating a few of the most important forms of equations, and par-

ticularly those of the second order. The subject of the present section may be divided under the following heads:

I. The integration of differential equations of the second order in the following cases:

1. Where they contain only the second differential coefficient and the independent variable.

2. Where they contain only the second differential coefficient and the dependant variable.

3. Where they contain the two differential coefficients and neither of the variables.

4. Where they contain the two differential coefficients and the independent variable.

5. Where they contain the two differential coefficients and the dependant variable.

6. Some of the more simple cases where they include both variables.

II. Some cases of the integration of equations of the nth order, which only contain differential coefficients and constants.

III. Certain cases of differential equations which include only one of the variables.

IV. Homogeneous equations of the first degree with respect to the dependant variable and its differentials.

V. Equations in general of the first degree with respect to y and its differentials.

I.

The integration of differential equations of the second order.

(352.) Let x be taken as the independent variable, and let the first and second differential coefficients be y', y''. The most general form for a differential equation of the second order is

$$\text{F}(xyy'y'') = 0.$$

We propose first to consider the five following cases of this,

$$
\begin{aligned}
&1.\ \text{F}(y''x) = 0,\\
&2.\ \text{F}(y''y) = 0,\\
&3.\ \text{F}(y''y') = 0,\\
&4.\ \text{F}(y''y'x) = 0,\\
&5.\ \text{F}(y''y'y) = 0.
\end{aligned}
$$

(353.) 1. By substituting $\frac{dy'}{dx}$ for y'', the first equation becomes

$$\text{F}(dy', dx, x) = 0,$$

which solved for dy', gives a result of the form

$$dy' = \text{X}dx,$$

where X is a function of x. This being an equation of the first order, may be integrated by the methods already explained, and its integral will be of the form

$$y' = \text{X}' + \text{A},$$

$$\text{or } dy = \text{X}'dx + \text{A}dx,$$

A being an arbitrary constant. This being again integrated, gives an equation of the form

$$y = \text{X}'' + \text{A}x + \text{B},$$

B being another arbitrary constant.

Thus, let $d^2y = ax^n dx^2$, $\therefore dy' = ax^n dx$,

$$\therefore y' = \frac{ax^{n+1}}{n+1} + \text{A}.$$

Substituting $\frac{dy}{dx}$ for y', we find

$$dy = \frac{ax^{n+1}dx}{n+1} + \text{A}dx, \therefore$$

$$y = \frac{ax^{n+2}}{(n+1)(n+2)} + \text{A}x + \text{B}.$$

(354.) 2. Let the form $\text{F}(y''y) = 0$ be supposed to be solved for y'', and therefore reduced to the form

$$\frac{d^2y}{dx^2} = \text{Y},$$

where Y is a function of y. Let both be multiplied by $2dy$ and integrated, and we find

$$\frac{dy^2}{dx^2} = 2\int \text{Y}dy + \text{A},$$

A being an arbitrary constant. The integral $\int \text{Y}dy$ may be determined by the established rules, and therefore the equation, after extracting the square root of both sides, assumes the form

$$\frac{dy}{dx} = \text{Y}', \ \therefore dx = \frac{dy}{\text{Y}'},$$

$$\therefore x = \int \frac{dy}{\text{Y}'},$$

which, when the integration of $\frac{dy}{\text{Y}}$ has been effected, becomes the integral of the sought equation.

Thus, let $a^2d^2y + ydx^2 = 0$, $\therefore$

$$\frac{d^2y}{dx^2} = -\frac{y}{a^2},$$

$$\therefore \frac{dy^2}{dx^2} = -\frac{y^2}{a^2} + \text{A},$$

$$\therefore \frac{dy}{dx} = \sqrt{\text{A} - \frac{y^2}{a^2}},$$

which is an equation of the first order.

(355.) 3. When the differential equation does not include either of the variables, it may immediately be reduced to an equation of the first order by substituting $\frac{dy'}{dx}$ for y'', by which it becomes $\text{F}\left(\frac{dy'}{dx}, y'\right) = 0$. This being solved for dx, assumes the form

$$dx = \text{F}(y')dy';$$

and since $y'dx = dy$, by multiplying both by y', we find

$$dy = y'\text{F}(y')dy'.$$

Hence by integration,

$$x = \int \text{F}(y')dy',$$

$$y = \int y'\text{F}(y')dy';$$

eliminating y' by these equations, the result, including two arbitrary constants, will be the integral sought.

Thus, let $d^2y = dx\sqrt{dy^2 + dx^2}$, $\therefore$

$$y'' = \sqrt{1 + y'^2},$$

$$\therefore dy' = \sqrt{1 + y'^2} \cdot dx,$$

$$\therefore dx = \frac{dy'}{\sqrt{1+y'^2}},$$

which may be integrated by the established rules. The value of y' being thence obtained as a function of x, the sought integral will be

$$y = \int y'dx.$$

(356.) 4. A differential equation of the second order of the form $\text{F}(y''y'x) = 0$ is reduced to one of the first order by substituting $\frac{dy'}{dx}$ for y''. The equation, therefore, assumes the form $\text{F}\left(\frac{dy'}{dx}, y', x\right) = 0$, which is an equation of the first order between y' and x. This being integrated by the established rules, gives an equation of the form $\text{F}(y'xc) = 0$, c being an arbitrary constant. Again, substituting $\frac{dy}{dx}$ for y', this becomes a differential equation of the first order between y and x. This being integrated as before, gives an equation of the form

$$\text{F}(xycc') = 0,$$

c, c' being the two arbitrary constants.

Thus, let $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y} = \frac{a^2}{2x}$. Hence

$$\frac{(1+y'^2)^{\frac{3}{2}}}{y''} = \frac{a^2}{2x}.$$

For y'' substitute $\frac{dy'}{dx}$, and we obtain

$$\frac{dy'}{(1+y'^2)^{\frac{3}{2}}} = \frac{2xdx}{a^2},$$

$$\therefore \frac{y'}{(1+y'^2)^{\frac{1}{2}}} = \frac{x^2+ac}{a^2},$$

$$\therefore y' = \frac{x^2+ac}{\sqrt{a^4-(x^2+ac)^2}},$$

$$\therefore y = \int \frac{(x^2+ac)dx}{\sqrt{a^4-(x^2+ac)^2}}.$$

The integral of which is obtained by the rules already established. This is the equation of the *elastic curve.*

(357.) In general, the equation $\text{F}(y'xc) = 0$ may be integrated by three different methods, which may be chosen according as they may severally be found best suited to the circumstances of the proposed equation.

1. If the equation admit of being solved for y', it may be reduced to the form

$$\frac{dy}{dx} = \text{X},$$

$$\therefore y = \int \text{X}dx.$$

2. If it admit of being solved for x, it may be reduced to the form

$$x = \text{F}(y').$$

But $y'dx = dy$, $\therefore y = \int y'dx = y'x - \int xdy'$. Hence

$$y = y'x - \int \text{F}(y')dy'.$$

The latter integral being determined, y' may be eliminated by means of this equation and $x = \text{F}(y')$, and the result will give the sought integral.

3. If the equation do not admit of solution for either x or

y', it will be necessary by a transformation to express x and y' as functions of a third variable z; let these functions be Z, Z', so that

$$x = \mathrm{Z}, \quad y' = \mathrm{Z}',$$
$$\therefore y = \int y'dx = \int \mathrm{Z}'dz,$$

which is the sought integral.

(358.) 4. If the equation have the form $\mathrm{F}(y''y'y) = 0$, it may be reduced to one of the first order, thus,

$$dy' = y''dx, \quad y'dx = dy,$$
$$\therefore y'dy' = y''dy, \; \therefore y'' = \frac{y'dy'}{dy}.$$

Let the proposed form be expressed thus,

$$y'' = f(y'y),$$
$$\therefore y'dy' = f(y'y)dy.$$

This being a differential equation of the first order between y' and y, may be integrated by the usual methods, and its integral will have the form $\mathrm{F}(y'yc) = 0$, c being the arbitrary constant. The integration of this presents two cases:

(359.) First, Where the variables may be separated, and therefore the equation may be reduced to either of the forms,

$$y' = \mathrm{Y},$$
$$y = \mathrm{Y}',$$

Y and Y' being functions of y and y' respectively.

In the first case, the integration is effected by reducing it to the form

$$dx = \frac{dy}{\mathrm{Y}},$$
$$\therefore x = \int \frac{1}{\mathrm{Y}} dy.$$

In the second case, since $dy = y'dx$,

$$\therefore y'dx = d\mathrm{Y}',$$

$$\therefore dx = \frac{d\text{Y}'}{y'},$$

$$\therefore x = \int \frac{d\text{Y}'}{y'}.$$

Eliminating y' by this and the equation $y = \text{Y}'$, the resulting equation will be the integral sought.

(360.) Secondly, If the variables y' and y cannot be separated, a transformation may be effected by expressing y' and y as functions, z', z, of another variable z. Since $y'dx = dy$, $\therefore z'dx = dz$, and the integration may be effected as in the last case of (357.).

(361.) 6. There are some remarkable cases in which differential equations of the second order, where they include both the variables, may be integrated without much difficulty. We shall consider the three following cases:

$$[1] \cdots \frac{d^2y}{dx^2} + \text{X}\frac{dy}{dx} + \text{X}'y = 0,$$

$$[2] \cdots \frac{d^2y}{dx^2} + \text{X}\frac{dy}{dx} + \text{X}'y + \text{X}'' = 0,$$

(where X, X', X'', are any functions of x.)

[3] $\cdots$ Where the equation is homogeneous with respect to the variables and their differentials.

(362.) 1. Let $y = e^{\int u dx}$, $\therefore \frac{dy}{dx} = ue^{\int u dx}$, and

$$\frac{d^2y}{dx^2} = e^{\int u dx}\left(u^2 + \frac{du}{dx}\right).$$

Making these substitutions, we find

$$\frac{du}{dx} + (u^2 + \text{X}u + \text{X}') = 0,$$

since the common factor $e^{\int u dx}$ disappears. This, being a differential equation of the first order, is integrated by the methods established in Section XVII.

(363.) 2. This equation may be reduced to the preceding by

any transformation which will remove the term x''. For this purpose, let $y = tz$. Hence

$$\frac{dy}{dx} = t\frac{dz}{dx} + z\frac{dt}{dx},$$

$$\frac{d^2y}{dx^2} = t\frac{d^2z}{dx^2} + 2\frac{dzdt}{dxdx} + z\frac{d^2t}{dx^2}.$$

Making these substitutions, we find

$$t\frac{d^2z}{dx^2} + 2\frac{dzdt}{dxdx} + z\frac{d^2t}{dx^2} + \text{x}\left(t\frac{dz}{dx} + z\frac{dt}{dx}\right)$$
$$+ \text{x}'tz + \text{x}'' = 0.$$

Let the variable z be limited by the condition

$$\frac{d^2z}{dx^2} + \text{x}\frac{dz}{dx} + \text{x}'z = 0 \cdot \cdot \cdot \cdot \cdot \cdot [1].$$

Hence the transformed equation, after dividing by z, becomes

$$\frac{d^2t}{dx^2} + \frac{dt}{dx}\left(\text{x} + \frac{2dz}{zdx}\right) + \frac{\text{x}''}{z} = 0 \cdot \cdot \cdot [2].$$

The first [1] of these equations may be integrated by the preceding article, and thence an equation found of the form $\text{F}(zx) = 0$. By this process $\frac{dz}{dx}$ will become known as a function of x, and thus the equation [2] will become integrable. The process in general may be conducted thus. Let

$$v = e^{\int \text{x}dx},$$

$$\therefore \text{x}dx = \frac{dv}{v}.$$

By this substitution, [2] becomes

$$\frac{d^2t}{dx} + \frac{dt}{dx}\left(\frac{dv}{v} + \frac{2dz}{z}\right) + \frac{\text{x}''dx}{z} = 0.$$

But since

$$\frac{dv}{v} + \frac{2dz}{z} = dl(vz^2) = \frac{d(vz^2)}{vz^2},$$

the equation may be reduced to the form

$$\frac{d^2t}{dx}vz^2 + \frac{dt}{dx}d(vz^2) + \text{x}''vzdx = 0.$$

Integrating it under this form, we find

$$\frac{dt}{dx}vz^2 + \int \text{x}''vzdx = 0,$$

$$\therefore t + \int \left\{ \frac{dx}{vz^2} \int (\text{x}''vzdx) \right\} = 0.$$

But since $t = \frac{y}{z}$, $\therefore$

$$y + z\int\left(\frac{dx}{vz^2}\int \text{x}''vzdx\right) = 0.$$

In this, v is a given function of x by the equation $v = e^{\int \text{x}dx}$, and z is a function of x by [1]. Hence this last equation is the integral sought. It will obviously include two arbitrary constants.

(364.) 3. If the equation be homogeneous with respect to y, x, dy, d^2y, and dx, let

$$y = ux, \quad \frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = \frac{z}{x},$$

u, y', and z, being considered as new variables. By this substitution, every term of the equation will have the same power of x as a factor, which being removed by division, the equation assumes the form

$$\text{F}(y'zu) = 0,$$

$$\text{or } z = f(y'u).$$

By differentiating $y = ux$, we find

$$dy = udx + xdu,$$

$$\therefore y'dx = udx + xdu,$$

$$\therefore \frac{dx}{x} = \frac{du}{y'-u}.$$

Also, since

$$\frac{d^2y}{dx^2} = \frac{z}{x},$$

$$\therefore x\frac{d^2y}{dx} = zdx,$$

$$\therefore xdy' = zdx,$$

$$\therefore \frac{dx}{x} = \frac{dy'}{z}.$$

Hence

$$\frac{dy'}{z} = \frac{du}{y'-u},$$

$$\therefore dy' - \frac{du}{y'-u}f(y'u) = 0.$$

This being an equation of the first order between y' and u, may be integrated by the rules for integrating such equations. The integration will give $y' = \mathrm{F}(u)$. Hence

$$\frac{dx}{x} = \frac{du}{\mathrm{F}(u)-u},$$

$$\therefore lx = \int\frac{du}{\mathrm{F}(u)-u}.$$

Eliminating u by this and $y = ux$, we obtain the integral of the proposed equation. It is obvious that this integral will include two arbitrary constants introduced by the two integrations effected prior to the elimination.

Thus, let $xd^2y = dydx$, $\therefore z = y'$, and hence

$$\frac{dy'}{y'} = \frac{du}{y'-u},$$

$$\therefore \tfrac{1}{2}y'^2 = \int(udy' + y'du) = y'u + \tfrac{1}{2}c.$$

But, also $\frac{dx}{x} = \frac{dy'}{y'}$, $\therefore x = ay'$. Eliminating y' by these equations, the result is $x^2 - 2axu = \mathrm{c}$. Eliminating u, we obtain the integral

$$x^2 - 2ay = \mathrm{c}.$$

II.

Integration of differential equations which do not contain either of the variables.

(365.) There are two cases in which differential equations, which do not include either variable, may be reduced to a form which is integrable by the rules for the integration of functions of a single variable. These two cases may be expressed thus:

$$1. \ldots \mathrm{F}\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}\right) = 0,$$

$$2. \ldots \mathrm{F}\left(\frac{d^n y}{dx^n}, \frac{d^{n-2}y}{dx^{n-2}}\right) = 0,$$

which denote any differential equations which include only two differential coefficients, the order of the one being in the first case lower by one, and in the second lower by two than the order of the other, and which exclude the variables.

(366.) 1. In the first case, let

$$\frac{d^{n-1}y}{dx^{n-1}} = u,$$

$$\therefore \frac{d^n y}{dx^n} = \frac{du}{dx}.$$

By which substitution, the form is changed to

$$\mathrm{F}\left(\frac{du}{dx}, u\right) = 0,$$

which being an equation of the first order may be integrated. This being effected, we shall obtain from the resulting equation

$$u = \mathrm{X},$$

$$\frac{d^{n-1}y}{dx^{n-1}} = \mathrm{X},$$

X being a function of x. The integration may be completed by Section XI.

(367.) 2. An equation of the form

$$F\left(\frac{d^n y}{dx^n}, \frac{d^{n-2}y}{dx^{n-2}}\right) = 0,$$

may be integrated by making the substitutions

$$u = \frac{d^{n-2}y}{dx^{n-2}},$$

$$\therefore \frac{d^2u}{dx^2} = \frac{d^n y}{dx^n}.$$

Hence the proposed equation becomes

$$F\left(\frac{d^2u}{dx^2}, u\right) = 0,$$

which being integrated by (354.), may be reduced to the form

$$u = X.$$

Whence

$$\frac{d^{n-2}y}{dx^{n-2}} = X;$$

this may be integrated by Section XI.

III.

Integration of differential equations which include but one of the variables.

(368.) Differential equations, which include only one of the variables, may be divided into two classes, those which include only the independent variable, and those which include only the dependant variable.

1°. The class of differential equations, which include only the independent variable, comes under the form

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0.$$

The exponent of the order of an equation of this kind may be always reduced by an easy and obvious transformation. Let

$$y' = \frac{dy}{dx}, \therefore \frac{dy'}{dx} = \frac{d^2y}{dx^2},$$

and, in general,

$$\frac{d^{n-1}y'}{dx^n} = \frac{d^ny}{dx^n}.$$

Hence the formula becomes

$$\text{F}\left(x, y', \frac{dy'}{dx}, \frac{d^2y'}{dx^2}, \ldots \frac{d^{n-1}y'}{dx^{n-1}}\right) = 0,$$

which is an equation of the $(n-1)$th order, including both variables y' and x.

2°. If the equation include the dependant variable only by the transformation (38.) for changing the independent variable, it may generally be reduced to the preceding case.

By this rule, an equation of the second order, when it includes only one of the variables, may be reduced to the first, one of the third to the second, and so on.

(369.) If an equation have the form

$$\text{F}\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^{n-2}y}{dx^{n-2}}\right) = 0;$$

it may by a similar process be reduced to an equation of the second order, including but one of the variables. For, let

$$u = \frac{d^{n-2}y}{dx^{n-2}},$$

$$\therefore \frac{du}{dx} = \frac{d^{n-1}y}{dx^{n-1}}, \frac{d^2u}{dx^2} = \frac{d^ny}{dx^n},$$

by which the equation becomes

$$\text{F}\left(\frac{d^2u}{dx^2}, \frac{du}{dx}, u\right) = 0,$$

which, by a similar process may be reduced to an equation of the first order, including two variables.

In general, therefore, a differential of the nth order, including no variable, may be reduced to one of the $(n-1)$th order, including one variable, or to one of the $(n-2)$th order, including two variables.

IV.

The integration of homogeneous equations of the first degree with respect to the dependant variable and its differentials.

(370.) Equations of this class come under the general formula

$$\frac{d^ny}{dx^n} + \text{A}\frac{d^{n-1}y}{dx^{n-1}} + \text{B}\frac{d^{n-2}y}{dx^{n-2}} \cdots\cdots \text{M}\frac{dy}{dx} + \text{N}y = 0 \cdots [1],$$

where A, B, $\cdots\cdots$ are functions of x, the independent variable.

Let $y = e^u$, and $\therefore$

$$dy = e^u du, \quad d^2y = e^u(d^2u + du^2),$$
$$d^3y = e^u(d^3u + 3d^2udu + du^3).$$

. . . .

By these substitutions, the proposed equation becomes divisible by e^u, and the resulting equation will be independent of y, and of the form

$$\frac{d^nu}{dx^n} + \text{A}\frac{d^{n-1}u}{dx^{n-1}} + \cdots\cdots \text{M}\frac{du}{dx} + \text{N} = 0^* \cdots\cdots [2]$$

which may be further reduced to an equation of the $(n-1)$th order, including both variables by (369.).

(371.) As it seldom happens that the equation [1] is integrable when its coefficients are variable, we shall at present consider the case only in which A, B, $\cdots\cdots$ are all constant quantities.

Thus, if the equation be

$$\frac{d^3y}{dx^3} + \text{A}\frac{d^2y}{dx^2} + \text{B}\frac{dy}{dx} + \text{C} = 0.$$

By the transformation already suggested, this becomes

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* The coefficients of this equation are not supposed to retain the same values as in [1], but are general representatives of functions of x.

$$\left.\begin{array}{r} d^3u + 3dud^2u + du^3 + \text{A}(d^2u + du^2)dx \\ + \text{B}dudx^2 + \text{C}dx^3 \end{array}\right\} = 0.$$

This equation may be reduced to one of the second order by substituting t for $\frac{du}{dx}$, by which it becomes

$$d^2t + (3t + \text{A})dtdx + (t^3 + \text{A}t^2 + \text{B}t + \text{C})dx^2 = 0.$$

Since A, B, C, are supposed constant, this equation may be satisfied by a constant value of t. For if t be supposed constant, $dt = 0$ and $d^2t = 0$, by which the equation is reduced to

$$t^3 + \text{A}t^2 + \text{B}t + \text{C} = 0.$$

In general there are three values of t, which are functions of A, B, C, and therefore constant, which will fulfil this equation. Let the three roots be

$$t = m_1, \quad t = m_2, \quad t = m_3.$$

And since

$$du = tdx, \quad \therefore u = tx + c, \quad \therefore y = e^{tx+c};$$

we obtain thus three values of y,

$$y_1 = e^{m_1x+c_1}, \quad y_2 = e^{m_2x+c_2}, \quad y_3 = e^{m_3x+c_3}.$$

Whence we have

$$y_1 = \text{C}_1e^{m_1x}, \quad y_2 = \text{C}_2e^{m_2x}, \quad y_3 = \text{C}_3e^{m_3x},$$

the three arbitrary constants being

$$e^{c_1}, \quad e^{c_2}, \quad e^{c_3}.$$

Since each of the equations between y and x just determined include but one arbitrary constant, they are only particular integrals. We shall, however, obtain the complete or general solution by equating the sum of the three particular values of y already found with y,

$$y = \text{C}_1e^{m_1x} + \text{C}_2e^{m_2x} + \text{C}_3e^{m_3x}.$$

There is no difficulty in proving that this equation satisfies the proposed differential-equation, for if it be differentiated, and its third differential obtained; the three constants being eliminated, the result will be identical with the proposed equation.

A principle much more extensive, however, may be established. Whatever be the nature of the coefficients A, B, C, it may be proved, that if y_1, y_2, y_3, be the three values of y, which separately satisfy the proposed equation, their sum $y_1 + y_2 + y_3$ being equated with y, will form an equation,

$$y = C_1y_1 + C_2y_2 + C_3y_3,$$

which will satisfy the proposed, and which, including three arbitrary constants, will be its general solution. If any of the particular values y_1, y_2, or y_3, contain an arbitrary constant, the corresponding multiplier may be omitted.

To prove this, let the last equation be thrice successively differentiated, substituting in the proposed equation the values of y, dy, d^2y, d^3y, and collecting together the multipliers of the same constant.

The result will be

$$\left.\begin{array}{l} C_1(d^3y_1 + Ad^2y_1dx + Bdy_1dx^2 + Cdx^3) \\ +C_2(d^3y_2 + Ad^2y_2dx + Bdy_2dx^2 + Cdx^3) \\ +C_3(d^3y_3 + Ad^2y_3dx + Bdy_3dx^2 + Cdx^3) \end{array}\right\} = 0.$$

Since we have supposed that y_1, y_2, y_3, severally satisfy the proposed equation, it is obvious that the preceding equation is fulfilled independently of C_1, C_2, C_3.

There will be no difficulty in extending this reasoning generally to the class of equations included in the general form [1], so that, if there be n particular values of y,

$$y_1, y_2, y_3 \cdots y_n,$$

which separately satisfy [1], its general solution is

$$y = C_1y_1 + C_2y_2 + C_3y_3 \cdots + C_ny_n.$$

When the coefficients A, B, C, are constant, the particular values of y are of the form $y = e^{mx}$, m being constant. Hence

$$dy = me^{mx}dx, \quad d^2y = m^2e^{mx}dx \cdots d^ny = m^ne^{mx}dx.$$

The proposed equation thus becomes divisible by e^{mx}, by which it is reduced to

$$m^n + \text{A}m^{n-1} + \text{B}m^{n-2} + \text{C}m^{n-3} \ldots \text{M}m + \text{N} = 0 \ldots [3].$$

Let the n roots of this equation be

$$m_1, m_2, m_3 \ldots m_n,$$

and we find the particular values of y corresponding to these severally,

$$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x} \ldots y_n = e^{m_n x},$$

and therefore the general solution is

$$y = \text{C}_1 e^{m_1 x} + \text{C}_2 e^{m_2 x} + \ldots \text{C}_n e^{m_n x} \ldots\ldots [4].$$

(372.) If any pair of values of m deduced from [3] be imaginary, they must have the forms $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, ∵ two of the *particular* integrals, deduced as above, assume the forms

$$y = \text{C}_1 e^{(a+b\sqrt{-1})x},$$

$$y = \text{C}_2 e^{(a-b\sqrt{-1})x}$$

and, therefore, their sum becomes

$$y = e^{ax}\left(\text{C}_1 e^{bx\sqrt{-1}} + \text{C}_2 e^{-bx\sqrt{-1}}\right).$$

But (256.),

$$e^{bx\sqrt{-1}} = \cos.bx + \sqrt{-1}\sin.bx,$$

$$e^{-bx\sqrt{-1}} = \cos.bx - \sqrt{-1}\sin.bx,$$

$$\because \text{C}_1 e^{bx\sqrt{-1}} + \text{C}_2 e^{-bx\sqrt{-1}} = (\text{C}_1 + \text{C}_2)\cos.bx + (\text{C}_1 - \text{C}_2)\sqrt{-1}\sin.bx.$$

Let the constants C_1, C_2, be so assumed, that $\text{C}_1 + \text{C}_2$ and $(\text{C}_1 - \text{C}_2)\sqrt{-1}$ shall be both real, which we are allowed to do since the differential equation is fulfilled independently of them; and let

$$\text{C}_1 + \text{C}_2 = p\sin.q,$$

$$(\text{C}_1 - \text{C}_2)\sqrt{-1} = p\cos.q,$$

p and q being arbitrary. Hence the corresponding terms of [4] become

$$y = e^{ax} \cdot p\sin.(bx + q);$$

and in the same manner, terms of the same form may be found for every pair of imaginary roots.

(373.) When the equation [3] has equal roots, the result is only a particular integral, for the corresponding terms of [4] become

$$c_1e^{mx} + c_2e^{mx} = (c_1 + c_2)e^{mx} = ce^{mx}.$$

There will be, therefore, one arbitrary constant less than the number necessary to give the integral its full generality.

In this case, therefore, the integral must be otherwise obtained.

(374.) The following process for obtaining the integral in this case was first proposed by *D'Alembert*.

Let the two equal roots be m_1 and m_2, and first let them be supposed to be unequal, and to differ by h, so that

$$m_2 = m_1 + h.$$

Hence the two corresponding terms of [4] would be

$$c_1e^{m_1x} + c_2e^{m_2x} = e^{m_1x}[c_1 + c_2e^{hx}].$$

But by (65.),

$$e^{hx} = 1 + \frac{hx}{1} + \frac{h^2x^2}{1.2} + \frac{h^3x^3}{1.2.3} + \ldots.$$

Let $c_1 + c_2 = E'$, and $c_2h = E''$. Hence

$$c_1e^{m_1x} + c_2e^{m_2x} = e^{m_1x}[E' + E''x + E''\frac{hx^2}{1.2} + \ldots\ldots],$$

where E' and E'' are arbitrary constants. As this will satisfy the proposed equation, whatever be the values of the arbitrary constants E', E'', and independent of h, we may suppose $h = 0$, which is equivalent to $m_1 = m_2$. This reduces the expression to

$$c_1e^{m_1x} + c_2e^{m_2x} = e^{m_1x}[E' + E''x],$$

which being substituted in [4], will render it a general solution, since it introduces the complete number of arbitrary constants.

It will be easy to extend the same process to the case

where three or more of the roots of [4] are equal. First, let $m_1 = m_2$, $\therefore$

$$y = e^{m_1 x}[E' + E''x] + C_3 e^{m_3 x} \cdots + C_n e^{m_n x}.$$

Let $m_3 = m_1 + h$, $\therefore$

$$e^{m_1 x}[E' + E''x] + C_3 e^{m_3 x} = e^{m_1 x}[E' + E''x + C_3 e^{hx}].$$

Developing e^{hx} as before, and substituting its value, we find the quantity within the parenthesis become

$$E' + C_3 + (E'' + C_3 h)x + C_3 \frac{h^2 x^2}{1.2} + C_3 \frac{h^3 x^3}{1.2.3} \cdots$$

Let $E' + C_3 = F'$, $E'' + C_3 h = F''$, $\frac{C_3 h^2}{1.2} = F'''$, and it becomes

$$F' + F''x + F'''x^2 + F''''. \frac{hx^3}{3} \cdots$$

In this, let $h = 0$, and $\because m_3 = m_1$, $\therefore$

$$e^{m_1 x}(E' + E''x) + C_3 e^{m_3 x} = e^{m_1 x}[F' + F''x + F'''x^2],$$

which being substituted in [4], renders the solution general as before. The same process may obviously be continued and applied to any number of equal roots.

V.

Linear equations of the first degree with respect to y *and its differentials.*

(375.) This class of equations are included under the formula

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + \cdots M\frac{dy}{dx} + Ny + X = 0.$$

Let the several coefficients A, B, $\cdots$ N be constant, and X any function of the independent variable x.

The integration of equations of this form is reduced to the resolution of algebraic equations, as in the last case, by

either of the following methods. The first is given by Euler in his Integral Calculus, and the other by Lagrange.

(376.) Let us first consider an equation of the second order,

$$\frac{d^2y}{dx^2} + \text{A}\frac{dy}{dx} + \text{B}y + \text{X} = 0.$$

Let $e^{-hx}dx$ be the factor which renders this equation integrable, and let $\int - \text{X}e^{-hx}dx = \text{X}' + c$. Hence the quantity

$$e^{-hx}\left(\frac{d^2y}{dx^2} + \text{A}\frac{dy}{dx} + \text{B}y\right)dx$$

must have an integral of the form

$$e^{-hx}\left(ay + b\frac{dy}{dx}\right).$$

To determine the arbitrary quantities h, a, b, let this be differentiated, and the result equated term for term with the former; ∴

$$-ha = \text{B}, \quad -hb + a = \text{A}, \quad b = 1,$$

$$\therefore h^2 + \text{A}h + \text{B} = 0, \quad a = -\frac{\text{B}}{h}, \quad b = 1.$$

The first equation gives h, and the last two a and b.

The immediate integral of the proposed equation is, therefore,

$$b\frac{dy}{dx} + ay = e^{hx}(\text{X}' + c).$$

If in this equation the two values of h determined by the equation $h^2 + \text{A}h + \text{B} = 0$ be successively substituted, and by the two equations thus found, $\frac{dy}{dx}$ be eliminated, the result will give the complete integral.

(377.) If the proposed equation be of the nth order, we may infer in the same manner, that the value of h is a root of the equation

$$h^n + \text{A}h^{n-1} + \cdots\cdots = 0.$$

And we shall have as many different immediate or first

integrals of the $(n-1)$th order as there are roots of this equation given.

If there be n roots given, by the n corresponding integrals, the $(n-1)$ differential coefficients may be eliminated, and the complete integral thus obtained; and if any number of roots less than the entire number be known, the order of the equation may be reduced by the elimination of as many differential coefficients.

(378.) We shall now explain Lagrange's method, which is founded upon the most general theorem which has yet been delivered upon the integration of differential equations.

In (371.) it was proved that the integral of the equation

$$\frac{d^n y}{dx^n} + \mathrm{A}\frac{d^{n-1}y}{dx^{n-1}} + \mathrm{B}\frac{d^{n-2}y}{dx^{n-2}} + \cdots \mathrm{M}\frac{dy}{dx} + \mathrm{N}y = 0 \quad [1]$$

was of the form

$$y = \mathrm{C}_1 e^{m_1 x} + \mathrm{C}_2 e^{m_2 x} + \mathrm{C}_3 e^{m_3 x} \quad \cdots\cdots \quad [2],$$

where $y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$, $\cdots\cdots$ were particular values of y which satisfied the equation [1], the sum of which, involved with the necessary number of arbitrary constants, constituted the general solution.

The equation to be integrated at present is more general than [1], being of the form

$$\frac{d^n y}{dx^n} + \mathrm{A}\frac{d^{n-1}y}{dx^{n-1}} + \mathrm{B}\frac{d^{n-2}y}{dx^{n-2}} \cdots \mathrm{M}\frac{dy}{dx} + \mathrm{N}y + \mathrm{X} = 0 \cdots [3],$$

X being any function of x. Let it then be proposed to assign the functions of x, into which the arbitrary constants C_1, C_2, $\cdots\cdots$ in [2] should be changed, in order that [2] should become the complete integral of [3]. If this can be effected, it will follow that the several terms of [2] will be so many particular values of y, which will satisfy the proposed equation [3], and, therefore, that if n particular values of y be given, the integral of the equation [3] may be immediately determined.

We shall investigate the values of the functions C_1, C_2,

. . . . in an equation of the third order, and the principle may thence be easily generalised.

Let the equation

$$y = c_1 y_1 + c_2 y_2 + c_3 y \quad . \quad . \quad . \quad . \quad . \quad [4]$$

be the sought integral, c_1, c_2, c_3, being arbitrary functions of x.

By differentiating, we obtain

$$dy = c_1 dy_1 + c_2 dy_2 + c_3 dy_3 + y_1 dc_1 + y_2 dc_2 + y_3 dc_3.$$

Let the arbitrary functions c_1, c_2, c_3, be limited by the condition

$$y_1 dc_1 + y_2 dc_2 + y_3 dc_3 = 0,$$

which reduces the differential equation to

$$dy = c_1 dy_1 + c_2 dy_2 + c_3 dy_3,$$

the form it would have had if c_1, c_2, c_3, were constant.

Differentiating this again, we find

$$d^2y = c_1 d^2y_1 + c_2 d^2y_2 + c_3 d^2y_3 + dc_1 dy_1 + dc_2 dy_2 + dc_3 dy_3.$$

Again, limiting the functions c_1, c_2, c_3, by the condition

$$dc_1 dy_1 + dc_2 dy_2 + dc_3 dy_3 = 0,$$

we find

$$d^2y = c_1 d^2y_1 + c_2 d^2y_2 + c_3 d^2y_3.$$

Differentiating this, the result will be

$$d^3y = c_1 d^3y_1 + c_2 d^3y_2 + c_3 d^3y_3 + dc_1 d^2y_1 + dc_2 d^2y_2 + dc_3 d^2y_3.$$

By substituting these values in the equation

$$\frac{d^3y}{dx^3} + A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy + X = 0 \quad . \quad . \quad . \quad . \quad [5],$$

we find

$$c_1\left\{\frac{d^3y_1}{dx^3} + A\frac{d^2y_1}{dx^2} + B\frac{dy_1}{dx} + Cy_1\right\}$$

$$+ c_2\left\{\frac{d^3y_2}{dx^3} + A\frac{d^2y_2}{dx^2} + B\frac{dy_2}{dx} + Cy_2\right\}$$

$$+ c_3\left\{\frac{d^3y_3}{dx^3} + A\frac{d^2y_3}{dx^2} + B\frac{dy_3}{dx} + Cy_3\right\}.$$

$$+\frac{dC_1d^2y_1}{dx^3}+\frac{dC_2d^2y_2}{dx^3}+\frac{dC_3d^2y_3}{dx^3}+X=0.$$

But since by hypothesis y_1, y_2, y_3, severally satisfy the equation

$$\frac{d^3y}{dx^3}+A\frac{d^2y}{dx^2}+B\frac{dy}{dx}+Cy=0 \cdot \cdot \cdot \cdot \cdot \cdot [6],$$

the former equation is reduced to

$$dC_1d^2y_1+dC_2d^2y_2+dC_3d^2y_3+Xdx^3=0.$$

By this equation, therefore, united with the conditions

$$y_1dC_1+y_2dC_2+y_3dC_3=0,$$
$$dy_1dC_1+dy_2dC_2+dy_3dC_3=0,$$

the values of the three differentials will be determined as functions of y_1, y_2, y_3, which being themselves determinate functions of x, we shall obtain by the methods for integrating functions of a single variable values of C_1, C_2, C_3, of the forms

$$C_1=X'+c_1,$$
$$C_2=X''+c_2,$$
$$C_3=X'''+c_3,$$

c_1, c_2, c_3, being arbitrary constants. Hence the equation [4] becomes

$$y=y_1(X'+c_1)+y_2(X''+c_2)+y_3(X'''+c_3),$$

which is the complete integral of the equation [5].

If two values only of y, which will satisfy the equation [6], be known, the integration of the proposed equation [5] will depend on that of an equation of the second order. For let the known values be y_1 and y_2, ∴

$$y=C_1y_1+C_2y_2,$$
$$\therefore dy=C_1dy_1+C_2dy_2,$$
$$y_1dC_1+y_2dC_2=0.$$

As only one of the functions C_1, C_2, is disposable, the equation

$$d^2y=C_1d^2y_1+C_2d^2y_2+dy_1dC_1+dy_2dC_2$$

cannot be further reduced, and by differentiation, it gives

$d^3y = c_1d^3y_1 + c_2d^3y_2 + 2d^2y_1dc_1 + 2d^2y_2dc_2 + dy_1d^2c_1 + dy_2d^2c_2$.
Making these substitutions in [5], it becomes, after multiplying by dx^3,

$$dy_1d^2c_1 + dy_2d^2c_2 + 2d^2y_1dc_1 + 2d^2y_2dc_2$$
$$+ \text{A}dy_1dc_1dx + \text{A}dy_2dc_2dx + \text{X}dx^3 = 0.$$

The differentials dc_2, d^2c_2, may be eliminated by this equation united with the equation

$$y_1dc_1 + y_2dc_2 = 0,$$

and its differential, and the resulting equation will only contain dc_1 and d^2c_1, and functions of x.

This equation is therefore reducible to a differential equation of the first order by (368.).

If only one value of y, which satisfies [6], be known, an auxiliary equation of the third order may be found, including dc_1 and d^2c_1, which may be reduced to one of the second order by (369.).

The method which we have just explained being extended to equations of every order, we conclude, that if n particular values of y satisfying the equation [1] be given, the general solution of this equation may immediately be obtained, and thence the general solution of the more general equation [3]. And further, that if $(n - 1)$ particular values of y only be given, that the integration of [3] may be reduced to the integration of an equation of the first order and first degree *.

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* See *Lacroix*, 4to, tom. ii. p. 529.

SECTION XXIV.

Praxis on the integration of equations of the second and superior orders.

In the arrangement of the examples on the integration of equations of the second and superior orders, we shall follow the order of the preceding section.

I.

Examples on equations of the second order.

(379.) 1. Equations of the form $F(y''x) = 0$.

Ex. 1. $d^2y = adx^2$. Hence we have $dy' = adx$, $\because$

$$y' = ax + c;$$

$$\because dy = axdx + cdx,$$

$$2y = ax^2 + 2cx + c',$$

c and c' being the two arbitrary constants.

Ex. 2. $\dfrac{ds^2}{dx^4} \cdot \dfrac{d^2y}{dx^2} = \dfrac{1}{a} \cos. \dfrac{x}{b}$, where

$$ds = \sqrt{dx^2 + dy^2} \text{ is constant,}$$

$$\because dsd^2s = dxd^2x + dyd^2y = 0;$$

and since

$$d^2x = -\frac{y'dy'dx}{1+y'^2},$$

where $y' = \dfrac{dy}{dx}$, $\because$

$$dy' = (1 + y'^2)\frac{d^2y}{dx} = \frac{ds^2d^2y}{dx^3},$$

$$\because dy' = \frac{dx}{a} \cos. \frac{x}{b},$$

$$\therefore y' = \frac{b}{a} \sin.\frac{x}{b} + c,$$

$$\therefore dy = \frac{bdx}{a} \sin.\frac{x}{b} + cdx,$$

$$\therefore y = -\frac{b^2}{a} \cos.\frac{x}{b} + cx + c',$$

c and c' being arbitrary constants.

Ex. 3. *A body moves uniformly along a given right line, and another moves uniformly in pursuit of it, to find the path of the latter.*

Let the given right line be the axis of x, and let yx be the co-ordinates of the place of the pursuer, and let c be the exponent of the ratio of the velocities of the two bodies. The pursuer may be considered at each instant as moving in the tangent to the curve of pursuit, and the tangent itself as continually passing through the pursued body.

The distance of the point where the tangent meets the axis of x from the origin is

$$\frac{xdy - ydx}{dy}.$$

Now, if s be the arc of the curve of pursuit measured from the point where the tangent is perpendicular to the axis of x, in which position we may assume the axis of y, we have

$$\frac{xdy - ydx}{dy} = cs.$$

Differentiating this equation, considering y as the independent variable, we find

$$-\frac{d^2x}{\sqrt{dy^2 + dx^2}} = \frac{cdy}{y}.$$

Let $x' = \dfrac{dx}{dy}$, $\therefore$

$$\frac{cdy}{y} = \frac{dx'}{\sqrt{1+x'^2}},$$

$$\therefore x' + \sqrt{1+x'^2} = (ay)^c,$$

where a is an arbitrary constant. Hence

$$x = \frac{a^c y^{c+1}}{2(c+1)} + \frac{1}{2a^c(c-1)y^{c-1}} + \text{c.}$$

This is the simplest case of curves of pursuit. See *Peacock*, p. 370.

(380.) 2. Equations of the form $\text{F}(y''y) = 0$.

Ex. 1. $a^2d^2y - ydx^2 = 0$, $\therefore$

$$y'' = \frac{y}{a^2};$$

multiplying by $2dy$, and integrating, we obtain

$$\frac{dy^2}{dx^2} = \frac{y^2+c}{a^2},$$

$$\therefore dx = \frac{ady}{\sqrt{y^2+c}},$$

$$\therefore x = al\,\frac{y+\sqrt{y^2+c}}{c'},$$

or

$$y = ce^{\frac{x}{a}} + c'e^{-\frac{x}{a}}.$$

Ex. 2. $d^2y\sqrt{ay} - dx^2 = 0$, $\therefore$

$$y'' = (ay)^{-\frac{1}{2}};$$

multiplying by $2dy$, and integrating,

$$\frac{dy^2}{dx^2} - 2\int(ay)^{-\frac{1}{2}}dy = 0,$$

$$\therefore \frac{dy}{dx} = \frac{\sqrt{4\sqrt{y}-c}}{\sqrt{a}},$$

which becomes

$$dx = \frac{\sqrt{a}}{\sqrt{4\sqrt{y}-c}}.dy,$$

which is integrated by the established rules.

(381.) 3. Equations coming under the form $F(y''y')=0$.

Ex. 1. Let $ad^2ydx + (dy^2 + dx^2)^{\frac{3}{2}} = 0$. Let $\frac{dy}{dx} = y'$,

$\therefore \frac{d^2y}{dx^2} = \frac{dy'}{dx}$. The equation, therefore, becomes

$$a\frac{dy'}{dx} + (1 + y'^2)^{\frac{3}{2}} = 0,$$

$$\therefore dx = -\frac{ady'}{(1+y'^2)^{\frac{3}{2}}};$$

and since $dy = y'dx$, $\therefore$

$$dy = -\frac{ay'dy'}{(1+y'^2)^{\frac{3}{2}}}.$$

Integrating these, we obtain

$$x = \text{A} - \frac{ay'}{(1+y'^2)^{\frac{1}{2}}}, \quad y = \text{B} + \frac{a}{(1+y'^2)^{\frac{1}{2}}}.$$

Eliminating y', we find

$$(\text{A} - x)^2 + (\text{B} - y)^2 = a^2.$$

This example proves that the circle is the only curve of which the radius of curvature is constant.

Ex. 2. $ad^2y = dydx$. By the usual substitution, we find

$$dx = \frac{ady'}{y'}, \quad \therefore x = aly' + c,$$

$$dy = ady', \quad \therefore y = ay' + c'.$$

Eliminating y', we find

$$x = c + al\left(\frac{y-c'}{a}\right).$$

Ex. 3. *To find the curve in which the radius of curvature varies as the angle under its tangent and the axis of* X.

Taking the arc s of the curve as the independent variable, the radius of curvature is

$$R = \frac{ds^2}{\sqrt{(d^2y)^2+(d^2x)^2}}.$$

But since $ds^2 = dy^2 + dx^2$, $\therefore$

$$dyd^2y + dxd^2x = 0.$$

Eliminating d^2y by this, we obtain

$$R = \frac{dsdy}{d^2x}.$$

Hence by hypothesis,

$$\frac{dsdy}{d^2x} = a \tan.^{-1}\frac{dy}{dx}.$$

But if $y' = \frac{dy}{dx}$, $\therefore ds = (1 + y'^2)^{\frac{1}{2}}dx$, $\therefore$ differentiating

$$d^2x \sqrt{1 + y'^2} + \frac{y'dxdy'}{\sqrt{1+y'^2}} = 0,$$

$$\therefore d^2x = -\frac{y'dxdy'}{1+y'^2}.$$

Hence the equation becomes

$$dx = -\frac{ady'}{(1+y'^2)^{\frac{3}{2}}}\tan.^{-1}y'.$$

Integrating this, we obtain

$$x = c - \frac{ay'}{\sqrt{1+y'^2}}\tan.^{-1}y' - \frac{a}{\sqrt{1+y'^2}},$$

$$y = c' + \frac{a}{\sqrt{1+y'^2}}\tan.^{-1}y' - \frac{ay'}{\sqrt{1+y'^2}}.$$

Eliminating y', we obtain the required curve.

Ex. 4. $\frac{dsdy}{d^2x} = \frac{adx}{dy}$, s being the independent variable, and

$$ds = \sqrt{dx^2 + dy^2},$$

$$\therefore d^2s = d^2x\sqrt{1 + y'^2} + \frac{y'dy'dx}{\sqrt{1+y'^2}},$$

$$\therefore d^2x = -\frac{y'dy'dx}{1+y'^2},$$

$$\therefore \frac{y' dx(1+y'^2)^{\frac{3}{2}}}{-y dy'} = \frac{a}{y'}.$$

Hence we obtain

$$x = c - \frac{a}{\sqrt{1+y'^2}} + a l \frac{1+\sqrt{1+y'^2}}{y'},$$

$$y = c' - \frac{a y'}{\sqrt{1+y'^2}}.$$

(382.) 4. Equations of the form $\text{F}(y'' y' x) = 0$.

Ex. 1. *To find the curve of which the radius of curvature varies inversely as the abscissa.*

By (137.),

$$\text{R} = -\frac{(1+y'^2)^{\frac{3}{2}}}{y''}.$$

Since R varies inversely as x, let

$$\text{R} = \frac{a^2}{2x},$$

a being constant. Hence the equation to be integrated is

$$a^2 y'' + 2x(1+y'^2)^{\frac{3}{2}} = 0.$$

This has been already integrated in (356.), and the result is the equation of the elastic curve. See *Poisson*, vol. i. p. 219.

Ex. 2. *To find the curve in which the radius of curvature is a given function of the abscissa.*

In this case

$$\frac{y'}{\sqrt{1+y'^2}} = \int \frac{dx}{\text{X}} = \text{X}',$$

$$\therefore y = \int \frac{\text{X}' dx}{\sqrt{1-\text{X}'^2}}.$$

This formula solves all the inverse problems relating to the radius of curvature.

Ex. 3. Let the given equation be

$$(1 + y'^2) - ay''(1 + y'^2)^{\frac{1}{2}} + xy'y'' = 0;$$

by the usual transformation, this may be reduced to

$$dx(1 + y'^2) + xy'dy' = ady'\sqrt{1 + y'^2},$$

dividing by $\sqrt{1 + y'^2}$, and integrating

$$x = \frac{ay' + b}{\sqrt{1 + y'^2}}.$$

But $y = y'x - \int xdy'$, $\therefore$

$$y = y'x - a\sqrt{1 + y'^2} - b\,.\,l[y' + \sqrt{1 + y'^2}] + blc,$$

$$\therefore y = \frac{by' - a}{\sqrt{1 + y'^2}} - bl.\frac{y' + \sqrt{1 + y'^2}}{c}.$$

By this and the former, y' being eliminated, we find

$$y = z - b \cdot l\frac{x + a}{c(b - z)},$$

where $z = \sqrt{a^2 + b^2 - x^2}$.

Ex. 4. $a(dx^2 + dy^2)^{\frac{3}{2}} = x^2dxd^2y$. Hence

$$y' = \frac{cx - a}{\sqrt{x^2 - (cx - a)^2}},$$

$$\therefore y = \int \frac{(cx - a)dx}{\sqrt{1 - c^2}\sqrt{x^2 + \frac{2acx}{1 - c^2} - \frac{a^2}{1 - c^2}}}.$$

$$\therefore y = c\sqrt{x^2 - (cx - a)^2} - \frac{a}{(1 - c^2)^{\frac{3}{2}}}l\left\{\frac{x(1 - c^2) + ac}{\sqrt{1 - c^2}} + \sqrt{x^2 - (cx - a^2)}\right\} + c'.$$

Ex. 5. $dx^3dy - xds^2d^2y = adxds\sqrt{d^2x^2 + d^2y^2}$, s being the independent variable, and

$$ds = \sqrt{dy^2 + dx^2}.$$

Hence we obtain $d^2s = 0$, $\therefore$

$$d^2x = -\frac{y'y''dx^2}{1+y'^2}, \quad d^2y = \frac{y''dx^2}{1+y'^2}.$$

Hence

$$y' - xy'' = ay'',$$

$$\text{or } y' - x\frac{dy'}{dx} = a\frac{dy'}{dx},$$

which comes under Clairaut's formula (350.).

Ex. 6. $adxdy^2 + x^2dxd^2y = nxdy\sqrt{dx^4 + a^2d^2y^2}$,

$$\therefore ay'^2dx + x^2dy' = nxy'\sqrt{dx^2 + a^2dy'^2}.$$

This being homogeneous with respect to x and y', let $x = y'u$, $\therefore$

$$\frac{dy'}{dx} = y'' = \frac{au^2 + nu\sqrt{1 - n^2a^2u^2 + a^2u^4}}{n^2a^2u^2 - 1},$$

and

$$\frac{dx}{x} = \frac{du}{y'' - u},$$

$$\therefore \frac{dx}{x} = \frac{du}{u} \cdot \frac{n^2a^2u^2 - 1}{1 + au - n^2a^2u^2 + n\sqrt{1 - n^2a^2u^2 + a^2u^4}}.$$

(383.) 5. Equations of the form $F(y''y'y) = 0$.

Ex. 1. $y''(yy' + a) = y'(1 + y'^2)$. Since $y''dy = y'dy'$, this is reduced to

$$dy'(yy' + a) = dy(1 + y'^2).$$

This being integrated by Clairaut's formula (350.), gives

$$y = ay' + c\sqrt{1 + y'^2},$$

$$x = \int\frac{dy}{y'} = al(by') + cl(y' + \sqrt{1 + y'^2}).$$

Eliminating y' by these equations, the integral may be found.

Ex. 2. Let the equation be

$$aby'' = \sqrt{y^2 + a^2y'^2};$$

this becomes, after substitution,

$$aby'dy' = dy\sqrt{y^2 + a^2y'^2}.$$

To integrate this, let $y = y'z$, and the equation becomes

$$abzdy - abydz = z^2 dy\sqrt{z^2 + a^2}.$$

The variables in this equation are separable by making $\sqrt{z^2 + a^2} = tz$, by which the values of z and dz being found, the equation is reduced to

$$\frac{dy}{y} = \frac{-btdt}{bt^2 - at - b},$$

which is integrable by rules already given.

Ex. 3. Let the equation be

$$y'' + \text{A}y' + \text{B}y = 0,$$

A and B being constant. This becomes, by the usual substitution,

$$y'dy' + \text{A}y'dy + \text{B}ydy = 0,$$

which being homogeneous, may be integrated by the substitution $y' = uy$. Hence

$$\frac{dy}{y} = \frac{-udu}{u^2 + \text{A}u + \text{B}} = \frac{-udu}{(u-a)(u-b)},$$

a and b being the roots of the equation

$$u^2 + \text{A}u + \text{B} = 0.$$

Also,

$$dx = \frac{dy}{y'} = \frac{dy}{uy} = \frac{-du}{(u-a)(u-b)},$$

$$\therefore \frac{dy}{y} - adx = \frac{-du}{u-b},$$

$$\frac{dy}{y} - bdx = \frac{-du}{u-a},$$

$$\therefore ly - ax = l\frac{m}{u-b},$$

$$ly - bx = l\frac{n}{u-a},$$

$$\therefore u - a = \frac{ne^{bx}}{y}, u - b = \frac{me^{ax}}{y}.$$

Hence

$$y(b - a) = ne^{bx} - me^{ax},$$

which is the complete integral.

This result may also be obtained by the process in (371.), which, when the roots a, b, are imaginary, gives

$$y = c'e^{mx}\cos.(hx + f);$$

and when they are equal, gives an integral of the form

$$y = ce^{ax}(x + k).$$

Ex. 4. $\dfrac{(dy^2 + y^2dx^2)^{\frac{3}{2}}}{2dy^2dx + y^2dx^3 - yd^2ydx} = ny$. Hence

$$\frac{(y'^2 + y^2)^{\frac{3}{2}}}{2y'^2 + y^2 - y''y} = ny,$$

$$\therefore dy(y'^2 + y^2)^{\frac{3}{2}} = 2ny'^2ydy + ny^3dy - ny^2y'dy'.$$

This being homogeneous with respect to y and y', let $y' = yu$,

$$\therefore \frac{dy}{y} = \frac{nudu}{(1+u^2)(n - \sqrt{1+u^2})},$$

$$dx = \frac{ndu}{(1+u^2)(n - \sqrt{1+u^2})}.$$

Ex. 5. $\dfrac{(y'^2 + y^2)^{\frac{3}{2}}}{2y'^2 + y^2 - y''y} = a$. Let $y'^2 + y^2 = z^2$, $\therefore$

$$y'dy' + ydy = y''dy + ydy = zdz,$$

$$\therefore y'' + y = \frac{zdz}{dy},$$

which gives

$$z^2dy = 2azdy - aydz,$$

$$\therefore y = \frac{cz}{2a - z} = \frac{c\sqrt{y^2 + y'^2}}{2a - \sqrt{y^2 + y'^2}},$$

which is an equation of the first order.

(384.) 6. Equations of the second order, which include both variables.

Ex. 1. $y'' + \dfrac{y'}{x} - \dfrac{y}{x^2} + \dfrac{a}{1 - x^2} = 0.$

Comparing this with the formula [2] (361.), we find

$$\text{X} = \frac{1}{x}, \quad \text{X}' = -\frac{1}{x^2}, \quad \text{X}'' = \frac{a}{1-x^2}.$$

Hence the equation [1] (363.), becomes

$$\frac{d^2z}{dx^2} + \frac{1}{x} \cdot \frac{dz}{dx} - \frac{z}{x^2} = 0,$$

which, by putting $z = e^{\int u dx}$, gives (362.),

$$\frac{du}{dx} + \left(u^2 + \frac{u}{x} - \frac{1}{x^2}\right) = 0.$$

This equation is rendered homogeneous by making $u = \frac{1}{u'}$; the variables are then separated by putting $x = su'$. Hence

$$\frac{du'}{u'} = -\frac{s^2+s-1}{s(s^2-1)}ds,$$

$$\therefore u' = \frac{1}{s}\sqrt{\frac{s+1}{s-1}},$$

neglecting the constant. Substituting for u' and s, their values, we find

$$u = \frac{x^2+1}{x(x^2-1)}, \int udx = l\frac{x^2-1}{x}, \; z = \frac{x^2-1}{x}.$$

Also,

$$v = e^{\int \text{X} dx} = e^{lx} = x.$$

Hence we find

$$\int \text{X}'' vzdx = \int adx = ax + b,$$

and, therefore,

$$y = \frac{x^2-1}{x}\int\frac{(ax+b)xdx}{(x^2-1)^2},$$

which is integrated by the rules for rational differentials.

II.

(385.) *Integration of equations which do not include either variable.*

Ex. 1. $\frac{a^2d^4y}{dx^4} = \frac{d^2y}{dx^2}$. Let $u = \frac{d^2y}{dx^2}$, $\therefore \frac{d^2u}{dx^2} = \frac{d^4y}{dx^4}$. Hence

the proposed equation becomes

$$\frac{a^2d^2u}{dx^2} = u.$$

Multiplying by $2du$, and integrating, we find

$$\frac{a^2du^2}{dx^2} = u^2 + b.$$

Hence

$$dx = \frac{adu}{\sqrt{u^2+b}},$$

$$\therefore x = al\left\{\frac{u+\sqrt{u^2+b}}{b'}\right\}.$$

And

$$\frac{dy}{dx} = \int udx = a\sqrt{u^2+b} + b''.$$

Hence we find

$$y = a^2u + ab''l[u + \sqrt{u^2+b}] + b''',$$

$$\therefore y = \frac{a^2b'}{2}e^{\frac{x}{a}} - \frac{ba^2}{2b'}e^{-\frac{x}{a}} + b''x + b''',$$

$$\text{or } y = ce^{\frac{x}{a}} + c'e^{-\frac{x}{a}} + c''x + c'''.$$

III, IV.

(386.) *Integration of equations, including* y *only*.

Ex. 1. Let the equation be

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

This equation being homogeneous with respect to y and its differentials, and of the first degree, comes under [1] (370.). By comparing the coefficients, we find

$$A = -2, \quad B = +2, \quad C = -2, \quad N = 1.$$

The equation [3] (371.) becomes, therefore,

$$m^4 - 2m^3 + 2m^2 - 2m + 1 = 0,$$

$$\text{or } (1-m)^2(1+m^2) = 0.$$

Hence its complete integral is

$$y = (a + bx)e^x + ce^{x\sqrt{-1}} + c'e^{x\sqrt{-1}},$$

$$\text{or } y = (a + bx)e^x + \text{A} \cos.x + \text{B} \sin.x.$$

SECTION XXV.

Of the integration of simultaneous differential equations of the first degree.

(387.) If m equations be given, involving $(m + 1)$ variables, all these variables, except one, may be considered as determinate functions of that one. The forms of these functions are determined by eliminating every combination of $(m - 1)$ variables, which can be obtained from the entire number of variables, except that one, on which the others are supposed to depend. This process will give m equations by which each of the m variables are connected with the independent variable, and by which they will be implicit functions of it. By the solution of these equations, they would become explicit functions of it.

If the equations between the several variables be differential equations, the process of elimination would be attended with considerable difficulty. Instead, therefore, of eliminating first, and then integrating the several differential equations, so as to obtain each variable as a function of the independent variable, we shall explain a method of integrating simultaneous differential equations without any previous elimination.

(388.) Let it be proposed to integrate simultaneously the two equations

$$\text{M}y + \text{N}x + \text{P}\frac{dy}{dt} + \text{Q}.\frac{dx}{dt} = \text{T},$$

$$\text{M}'y + \text{N}'x + \text{P}'\frac{dy}{dt} + \text{Q}'\frac{dx}{dt} = \text{T}',$$

which are the most general equations of the first degree between the variables x, y, and the differential coefficients $\frac{dy}{dt}$ and $\frac{dx}{dt}$. In these equations the several coefficients $\text{M}, \text{M}', \text{N}, \text{N}', \cdots$ are supposed to be functions of the independent variable t.

Let these equations be expressed thus,

$$(\text{M}y + \text{N}x)dt + \text{P}dy + \text{Q}dx = \text{T}dt,$$
$$(\text{M}'y + \text{N}'x)dt + \text{P}'dy + \text{Q}'dx = \text{T}'dt.$$

Multiplying the second by an arbitrary function (θ) of t, and adding the product to the first, we obtain the equation

$$\text{H}ydt + \text{K}xdt + \text{R}dy + \text{S}dx = \text{U}dt,$$

where

$$\text{H} = \text{M} + \text{M}'\theta, \qquad \text{K} = \text{N} + \text{N}'\theta,$$
$$\text{R} = \text{P} + \text{P}'\theta, \qquad \text{S} = \text{Q} + \text{Q}'\theta,$$
$$\text{U} = \text{T} + \text{T}'\theta.$$

This equation may be expressed under the form

$$\text{H}(y + \frac{\text{K}}{\text{H}}x)dt + \text{R}(dy + \frac{\text{S}}{\text{R}}dx) = \text{U}dt.$$

This will become a linear equation of the first order with respect to $y + \frac{\text{H}}{\text{K}}x$ and $d(y + \frac{\text{H}}{\text{K}}x)$, if

$$dz = dy + \frac{\text{S}}{\text{R}} \cdot dx,$$

where $z = y + \frac{\text{H}}{\text{K}}x$; for in that case we have

$$dz + \frac{\text{H}}{\text{R}}zdt = \frac{\text{U}}{\text{R}} \cdot dt,$$

which is of the form integrated in (314.).

The condition

$$d(y + \frac{K}{H}x) = dy + \frac{S}{R}dx,$$

gives

$$d\left(\frac{K}{H}x\right) = \frac{S}{R}dx,$$

$$\therefore \frac{K}{H}dx + xd\frac{K}{H} = \frac{S}{R}dx,$$

$$\therefore \left(\frac{K}{H} - \frac{S}{R}\right)dx + xd\left(\frac{K}{H}\right) = 0,$$

$$\therefore d\frac{K}{H} = 0, \quad \frac{K}{H} = \frac{S}{R}.$$

Substituting for K, H, S, and R, their values, and differentiating and eliminating θ, the resulting equation between the coefficients M, M′, ... will be the condition under which the integration of the proposed equations can be effected by the formula (314.).

(389.) The simultaneous integration of the equations

$$(My + Nx)dt + Pdy + Qdx = Tdt,$$
$$(M'y + N'x)dt + P'dy + Q'dx = T'dt,$$

may also be effected thus: let dy and dx be alternately eliminated, and the results will be two equations of the forms

$$dy + (Py + Qx)dt = Tdt,$$
$$dx + (P'y + Q'x)dt = T'dt,$$

the coefficients representing the functions of the former coefficients, which are determined by the process of elimination. Multiplying, as before, the second by θ, and adding, we find

$$dy + \theta dx + [(P + P'\theta)y + (Q + Q'\theta)x]dt = (T + T'\theta)dt.$$

Let $y + \theta x = z$, $\therefore$

$$dy + \theta dx = dz - xd\theta, \quad y = z - \theta x.$$

By these substitutions, the equation becomes

$$dz+(P+P'\theta)zdt-x[d\theta+[(P+P'\theta)\theta-(Q+Q'\theta)]dt]=(T+T'\theta)dt.$$

Let such a value be assigned to the function θ as will satisfy the equation

$$d\theta + [(\text{P} + \text{P}'\theta)\theta - (\text{Q} + \text{Q}'\theta)]dt = 0;$$

and the equation will be reduced to the form

$$dz + \text{T}zdt = \text{T}'dt,$$

T and T$'$ expressing new functions of t. This form is integrated as before by (314.).

(390.) If the coefficients P, P$'$, ... instead of being functions of t, as we before supposed, be constant quantities, we have the conditions

$$d\theta = 0, \quad (\text{P} + \text{P}'\theta)\theta - (\text{Q} + \text{Q}'\theta) = 0.$$

The function θ then becomes a constant quantity, and its values are the roots of the latter equation. Let these be θ', θ''. The equation

$$dz + (\text{P} + \text{P}'\theta)zdt = (\text{T} + \text{T}'\theta)dt$$

becomes

$$dz + mzdt = \text{V}dt,$$

by putting

$$m = \text{P} + \text{P}'\theta, \quad \text{V} = \text{T} + \text{T}'\theta.$$

The integral of which is, (314.),

$$z = e^{-mt}\int e^{mt}\text{V}dt.$$

Whence we deduce

$$y + \theta'x = e^{-m't}\int e^{m't}\text{V}'dt,$$
$$y + \theta''x = e^{-m''t}\int e^{m''t}\text{V}''dt,$$

by substituting successively the two values of θ, and the corresponding values of m and V.

(391.) We shall now apply the same principles to two differential equations between three variables. These may by alternate elimination be, as before, reduced to the forms

$$du + (\text{P}u + \text{Q}x + \text{R}y)dt = \text{T}dt,$$
$$dx + (\text{P}'u + \text{Q}'x + \text{R}'y)dt = \text{T}'dt,$$
$$dy + (\text{P}''u + \text{Q}''x + \text{R}''y)dt = \text{T}''dt.$$

Multiply the second by θ, and the third by θ', and add

the results to the first. In the equation resulting from this process let

$$u + \theta x + \theta' y = z,$$

$$\therefore du + \theta dx + \theta' dy = dz - xd\theta - yd\theta',$$

$$u = z - \theta x - \theta' y.$$

By these substitutions, an equation being obtained, let the coefficients of x and y in it be supposed to become $= 0$ by the values of the arbitrary quantities θ, θ'. This gives the equations

$$\frac{d\theta}{dt} = Q + Q'\theta + Q''\theta' - (P + P'\theta + P''\theta')\theta,$$

$$\frac{d\theta'}{dt} = R + R'\theta + R''\theta' - (P + P'\theta + P''\theta')\theta',$$

for the determination of θ. These conditions reduce the proposed equation to

$$dz + (P + P'\theta + P''\theta')zdt = (T + T'\theta + T''\theta')dt.$$

Substituting in this values of θ which satisfy the former equations, it will be integrable as in the former case.

If the several coefficients P, P', ... be constant quantities, we have $d\theta = 0$, $d\theta' = 0$, $\therefore$

$$(P + P'\theta + P''\theta')\theta = Q + Q'\theta + Q''\theta',$$

$$(P + P'\theta + P''\theta')\theta' = R + R'\theta + R''\theta',$$

which, by putting $P + P'\theta + P''\theta' = m$, become

$$(m - Q')\theta - Q''\theta' = Q,$$

$$(m - R'')\theta' - R'\theta = R.$$

These equations will give values for θ, θ', which being substituted in the value of m, will give an equation of the third degree to determine m. Each of the roots of this equation gives corresponding values for θ and θ'. If then we put $T + T'\theta + T''\theta' = V$, we obtain three systems of values for θ, θ', m, V; scil.

$$\theta_1, \theta'_1, m_1, V_1, \quad \theta_2, \theta'_2, m_2, V_2, \quad \theta_3, \theta'_3, m_3, V_3,$$

which being successively substituted in

$$z = e^{-mt}\int e^{mt}\mathrm{v}dt,$$

give

$$u + \theta_1 x + \theta'_1 y = e^{-m_1 t}\int e^{m_1 t}\mathrm{v}_1 dt,$$
$$u + \theta_2 x + \theta'_2 y = e^{-m_2 t}\int e^{m_2 t}\mathrm{v}_2 dt,$$
$$u + \theta_3 x + \theta'_3 y = e^{-m_3 t}\int e^{m_3 t}\mathrm{v}_3 dt.$$

It is unnecessary to pursue this process to a greater number of equations, as it is very easily generalised. We shall not enter here into an examination of the consequences of the values of θ, θ', becoming imaginary or equal, as it would protract the discussion to an undue length.

The same principles are applicable to equations of superior orders, by reducing them to equations of the first order.

SECTION XXVI.

The integration of equations by approximation.

(392.) When a differential equation cannot be integrated in finite terms by any known methods, it becomes necessary to approximate to the value of the differential coefficient by a series. A method has already been explained, by which the integral may be obtained in a series of ascending integral powers of x. But it sometimes happens that the nature of the functions engaged in the equation is such, that this form of development is inapplicable. In such cases, the form of the series must be obtained by other analytical contrivances.

If the form of the series for y in powers of x be

$$y = \mathrm{A}x^a + \mathrm{B}x^b + \mathrm{C}x^c + \cdots$$

the problem is reduced to the determination of the exponents $a, b, c, \cdots$ and the coefficients $\mathrm{A}, \mathrm{B}, \mathrm{C}, \cdots$ so as to satisfy the proposed differential equation. To effect

this, let the values of dy, d^2y, d^3y, ... be derived from the series and equated with the same quantities derived from the proposed equation. The several results should be identical, and therefore the coefficients of the corresponding dimensions of the variable should be equal. The values of the coefficients and exponents will, in general, be derived from these conditions.

A few examples will render these general principles easily comprehended.

(393.) Ex. 1. Let the proposed equation be

$$(dx + dy)y = dx;$$

and let the series be

$$y = \text{A}x^a + \text{B}x^b + \text{C}x^c \cdots\cdots$$

the exponents being in ascending order.

By differentiating, we find

$$dy = a\text{A}x^{a-1}dx + b\text{B}x^{b-1}dx + c\text{C}x^{c-1}dx \cdots\cdots$$

Making this substitution for dy in the given equation, and expunging the common factor dx, we find

$$(1 + \text{A}ax^{a-1} + \text{B}bx^{b-1} + \cdots\cdots)(\text{A}x^a + \text{B}x^b + \text{C}x^c \cdots\cdots) = 1.$$

Hence

$$\left.\begin{array}{llll} \text{A}^2ax^{2a-1} + \text{AB}(a+b)x^{a+b-1} & + \text{AC}(a+c)x^{a+c-1} & + \cdots\cdots \\ -1 + \text{A}x^a & + \text{B}^2bx^{2b-1} & + \cdots\cdots \\ & + \text{B}x^b & + \cdots\cdots \end{array}\right\} = 0.$$

This will be rendered identically 0 by the following conditions,

$$2a - 1 = 0, \quad a + b - 1 = a, \quad a + c - 1 = b \cdots\cdots$$

$$\text{A}^2a = 1; \quad \text{AB}(a + b) + \text{A} = 0 \cdots\cdots$$

Hence

$$a = \frac{1}{2}, \quad b = 1, \quad c = \frac{3}{2} \cdots\cdots$$

$$\text{A} = \sqrt{2}, \quad \text{B} = -\frac{2}{3}, \quad \text{C} = \frac{\sqrt{2}}{18},$$

$$\therefore y = \sqrt{2}\, x^{\frac{1}{2}} - \frac{2}{3}x^{\frac{2}{2}} + \frac{\sqrt{2}}{18}x^{\frac{3}{2}} - \cdots\cdots$$

If the law of the exponents $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, ... had been known

in the first instance, the coefficients might have been immediately deduced, or the series of Maclaurin might have been immediately applied by substituting x^2 for x.

Ex. 2. Let the equation be

$$dy + ydx = mx^n dx;$$

let

$$y = \text{A}x^a + \text{B}x^{a+1} + \text{C}x^{a+2} \dots .$$

Differentiating this, and substituting the values of y and dy in the proposed equation, and omitting the factor dx, the result arranged by the dimensions of x is

$$\left.\begin{array}{llll} a\text{A}x^{a-1} + (a+1)\text{B}x^a + (a+2)\text{C}x^{a+1} + (a+3)\text{D}x^{a+2} + \dots \\ -mx^n + \qquad \text{A}x^a + \qquad \text{B}x^{a+1} + \qquad \text{C}x^{a+2} + \dots \end{array}\right\} = 0.$$

This is rendered identically 0 by

$$n = a - 1, \because a = n + 1, \text{A} = \frac{m}{a}, \text{B} = -\frac{m}{a(a+1)},$$

$$\text{C} = \frac{m}{a(a+1)(a+2)}, \quad \text{D} = \frac{-m}{a(a+1)(a+2)(a+3)}.$$

Hence we obtain

$$y = mx^n \left\{ \frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \right\}$$

the law of which is evident. This series is, however, not the complete integral, unless the arbitrary constant be introduced. This is always the case when the arbitrary constant in the development of y in powers of x cannot be separated from x.

We may, however, obtain the complete integral in the following manner. Let $\text{F}(xyc) = 0$ be the integral sought. To determine the constant c, it would be necessary to find some one system of values of the variables x, y, which will satisfy the primitive equation. Suppose a, b, be such a system. The condition $\text{F}(a, b, c) = 0$, would give c in terms of a and b. Let the expression for y, derived from the differential equation, be prepared in such a manner, that when x becomes equal to a, y will necessarily be equal

to b. This may be done by substituting $z + a$ for x, and $u + b$ for y, and then developing u in a series of powers of z, so that u and z should $= 0$ at the same time; then substituting $x - a$ for z, and $y - b$ for u. Under these circumstances, the resulting condition would give $x = a$ and $y = b$ at the same time, and the quantities a and b would supply the place of the arbitrary constant. The integral would therefore have all the necessary generality.

The proposed equation

$$dy + ydx = mx^n dx$$

becomes, by the transformation just explained,

$$du + (b + u)dz = m(a + z)^n dz.$$

Let

$$u = \text{A}z^\alpha + \text{B}z^{\alpha+1} + \text{C}z^{\alpha+2} + \ldots.$$

Hence we obtain

$$\left.\begin{array}{l} \alpha\text{A}z^{\alpha-1} + (\alpha + 1)\text{B}z^\alpha + (\alpha + 2)\text{C}z^{\alpha+1} + \ldots \\ + b \quad + \qquad \text{A}z^\alpha + \qquad \text{B}z^{\alpha+1} + \ldots \\ - ma^n - m\frac{n}{1} a^{n-1}z - m\frac{n.n-1}{1.2} a^{n-2}z^2 - \ldots \end{array}\right\} = 0.$$

The condition $\alpha - 1 = 0$, gives

$$\alpha = 1, \ \text{A} = ma^n - b, \ \text{B} = \frac{mna^{n-1} - ma^n + b}{1.2},$$

$$\text{C} = \frac{mn(n-1)a^{n-2} - mna^{n-1} + ma^n - b}{1.2.3}, \text{ \&c.}$$

The investigation may also be conducted by Taylor's series. If b be considered as a function of a, it will become

$$b + \frac{db}{da} \cdot \frac{z}{1} + \frac{d^2b}{da^2} \cdot \frac{z^2}{1.2} + \frac{d^3b}{da^3} \cdot \frac{z^3}{1.2.3} + \ldots$$

when a is changed into $a + z$. And since $y = b + u$, when $x = a + z$, $\therefore$

$$y = b + \frac{db}{da} \cdot \frac{z}{1} + \frac{d^2b}{da^2} \cdot \frac{z^2}{1.2} + \ldots$$

Since a and b are a system of values of x and y, which

satisfy the proposed equation, the same relation must subsist between a, b, and $\frac{db}{da}$, as between x, y, and $\frac{dy}{dx}$. Hence the value of $\frac{db}{da}$ will be found by substituting for x and y in the differential equation the values a and b, and thence deriving $\frac{db}{da}$. The successive differentials of this equation will, therefore, give the values of $\frac{d^2b}{da^2}$, $\frac{d^3b}{da^3}$, ...

When z is small, the series will converge rapidly. To extend numerical calculations to greater values of x, it will be necessary to substitute successively a_1 for $a + z$, and to change x into $a_1 + z$, and then substitute a_2 for $a_1 + z$, and $a_2 + z$ for x, and so on.

This process becomes inapplicable when any differential coefficient becomes infinite. This can only happen when $x = a$ renders y infinite, or when the development of y contains fractional powers of x. If the series of exponents be known in this case, we may frequently employ Taylor's series. If the exponents be such, that they are all multiples of any one fraction $\frac{m}{n}$, then let $x = z^{\frac{m}{n}}$, and the series of Taylor will be applicable.

Ex. 3. Let the proposed equation be

$$d^2y + cx^n y dx^2 = 0.$$

Let

$$y = \text{A}x^a + \text{B}x^{a+h} + \text{C}x^{a+2h} \dots .$$

$$\begin{aligned} \therefore d^2y = [&a(a-1)\text{A}x^{a-2} \\ &+ (a+h)(a+h-1)\text{B}x^{a+h-2} \\ &+ (a+2h)(a+2h-1)\text{C}x^{a+2h-2} + \dots]dx^2. \end{aligned}$$

Also,

$$-cx^n y = \frac{d^2y}{dx^2} = -c[\text{A}x^{a+n} + \text{B}x^{a+h+n} + \text{C}x^{a+2h+n} \dots].$$

It is obvious, that it is impossible that the terms

$$a(a-1)\text{A}x^{a-2}, \quad -c\text{A}x^{a+n},$$

can be identified in any other case than that in which $n = -2$, which would only include a particular case of the proposed equation. Therefore, such a value must be assigned to a as will remove the first term altogether. This is effected either by

$$a = 0, \text{ or } a = 1.$$

We may then identify the terms

$$(a+h)(a+h-1)\text{B}x^{a+h-2}, \quad -c\text{A}x^{a+n},$$

by the condition

$$h - 2 = n, \because h = n + 2.$$

The two series will then agree, and the coefficients will be determined by the equations

$$(a+\ \ h)(a+\ \ h-1)\text{B} + c\text{A} = 0,$$
$$(a+2h)(a+2h-1)\text{C} + c\text{B} = 0.$$

. . . .

. . . .

Since the number of arbitrary quantities A, B, ... is greater by one than the number of equations, one (A) will remain arbitrary. If the two values of a already obtained be substituted successively for it, we find the two series

$$\text{A} - \frac{\text{A}cx^{n+2}}{(n+1)(n+2)} + \frac{\text{A}c^2x^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)}$$

$$- \frac{\text{A}c^3x^{3n+6}}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)} \cdots\cdot$$

$$\text{A}x - \frac{c\text{A}x^{n+3}}{(n+2)(n+3)} + \frac{c^2\text{A}x^{2n+5}}{(n+2)(n+3)(2n+4)(2n+5)}$$

$$- \frac{c^3\text{A}x^{3n+7}}{(n+2)(n+3)2n+4)(2n+5)(3n+6)(3n+7)} \cdots\cdot$$

Each of these series are particular integrals, since each contains only one arbitrary constant; but by changing the A in the last series into A′, and adding them, the result will be the

complete integral, since the proposed equation is homogeneous with respect to y and d^2y.

(394.) Another method of approximating to the integrals of equations by a continued fraction merits attention.

Let

$$y = \cfrac{\text{A}x^a}{1 + \cfrac{\text{B}x^b}{1 + \cfrac{\text{C}x^c}{1 + \cfrac{\text{D}x^d}{1 + \cdots}}}}$$

where the coefficients and exponents are indeterminate.

Let $\text{A}x^a$ and $a\text{A}x^{a-1}dx$ be first substituted for y and dy in the proposed differential equation. If the integral corresponding to an indefinitely small value of the independent variable be sought, let all the terms of this equation involving the powers of x, whose exponents exceed the lowest exponent, be rejected.

By a comparison of the corresponding terms, the values of A and a may be determined. Next let

$$\frac{\text{A}x^a}{1+\text{B}x^b}$$

be substituted for y, and its differential for dy, and by a similar process, B and b may be determined, and this process may be continued until a sufficiently near approximation to the integral may be found.

Ex. 1. Let the proposed equation be

$$mydx + (1 + x)dy = 0.$$

First substitute $\text{A}x^a$ for y, and $a\text{A}x^{a-1}dx$ for dy. Hence we find

$$(m + a)\text{A}x^{a} + a\text{A}x^{a-1} = 0,$$

$$\therefore (m + a)\text{A}x + a\text{A} = 0.$$

Neglecting the term $\text{A}x$, we find

$$a\text{A} = 0; \therefore a = 0,$$

the quantity A remaining arbitrary. Now let

$$y = \frac{\text{A}}{1 + \text{B}x^b}$$

be substituted in the proposed equation, and the result is

$$m(1 + \text{B}x^b)dx - (1 + x)d(\text{B}x^b) = 0,$$

$$\therefore m - b\text{B}x^{b-1} + (m - b)\text{B}x^b = 0.$$

Rejecting the last term, we find

$$m = b\text{B}x^{b-1},$$

which is satisfied by

$$b = 1, \quad \text{B} = m.$$

Now let

$$y = \cfrac{\text{A}}{1 + \cfrac{mx}{1 + \text{C}x^c}}$$

Substituting this for y, and its differential for dy, as before, we obtain $c = 1$ and $\text{C} = -\frac{m-1}{2}$, and by continuing the process, we find

$$y = \cfrac{\text{A}}{1 + \cfrac{mx}{1 - (m-1)\cfrac{\frac{x}{2}}{1 + \frac{1}{3}(m+1)\cdot\cfrac{\frac{x}{2}}{1 - \frac{1}{3}(m-2)\cdot\cfrac{\frac{x}{2}}{1 + \frac{1}{5}(m+2)\cdot\cfrac{\frac{x}{2}}{1 + \ldots}}}}}}$$

(395.) When, as is frequently the case, the integral of the proposed equation can also be obtained in finite terms, this method furnishes a mean for converting the function which expresses the integral into a continued fraction. Hence, to convert a function of x into a continued fraction, differentiate it, and integrate the result by the continued fraction, supplying the arbitrary constant; this fraction will represent the proposed function. Thus, in the example just given, the integral in finite terms is $A(1+x)^{-m}$. Hence this function is equivalent to the continued fraction already found, and dividing both by the arbitrary constant A, we find

$$(1+x)^m = 1 + \cfrac{mx}{1 - \cfrac{\frac{1}{2}(m-1)x}{1 + \cfrac{\frac{1}{3}(m+1)\cdot\frac{x}{2}}{1 - \cfrac{\frac{1}{3}(m-2)\cdot\frac{x}{2}}{1 + \cfrac{\frac{1}{5}(m+2)\cdot\frac{x}{2}}{1 + \cdots\cdots \atop \cdots\cdots\cdots}}}}}$$

By comparing the developments of e^x with that of $\left(1+\frac{x}{m}\right)^m$, we find that they become identical when m is supposed infinite. Hence, if in the fraction just found, $\frac{x}{m}$ be substituted for m, and, in the result, m be supposed infinite, we obtain

$$e^x = 1 + \cfrac{x}{1 - \cfrac{\frac{1}{1}\cdot\frac{x}{2}}{1 + \cfrac{\frac{1}{3}\cdot\frac{x}{2}}{1 - \cfrac{\frac{1}{3}\cdot\frac{x}{2}}{1 + \cfrac{\frac{1}{5}\cdot\frac{x}{2}}{1 - \ldots\ldots}}}}}$$

Ex. 2. Let the proposed equation be

$$dx - (1 + x^n)dy = 0.$$

By a similar process to that used in the former example, we obtain

$$y = \int\frac{dx}{1+x^n} = \cfrac{x}{1 + \cfrac{\frac{x^n}{n+1}}{1 + \cfrac{\frac{n^2x^n}{(n+1)(2n+1)}}{1 + \cfrac{\frac{(n+1)^2x^n}{(2n+1)(3n+1)}}{1 + \cfrac{\frac{(2n)^2x^n}{(3n+1)(4n+1)}}{1 + \ldots\ldots}}}}}$$

In this case, if $n = 1$, we find

$$l(1+x) = \cfrac{x}{1+\cfrac{\frac{x}{2}}{1+\cfrac{\frac{x}{2.3}}{1+\cfrac{\frac{2x}{3.2}}{1+\cfrac{\frac{2x}{5.2}}{1+\cdots}}}}}$$

If $n = 2$, we find

$$\tan.^{-1}x = \cfrac{x}{1+\cfrac{\frac{x^2}{1.3}}{1+\cfrac{\frac{4x^2}{3.5}}{1+\cfrac{\frac{9x^2}{5.7}}{1+\cdots}}}}$$

There are other methods of approximation, one depending on the method of substitutions used by Newton, to resolve by approximation algebraic equations, combined with the methods of integrating equations of the first degree; also one derived from *Lagrange*'s theory of the variation of arbitrary constants; but the discussion of these would lead us into details unsuitable to the ends of this treatise.

SECTION XXVII.

Integration of differential equations of two variables by the geometry of plane curves.

(396.) Before the methods of approximation to the roots of algebraic equations were known, a method of representing them by the co-ordinates of the intersections of plane curves was used. (Geometry, Sec. XX.) This method is, however, now introduced into the elements of mathematical science only on account of its elegance, since it has been altogether superseded, for practical purposes, by the more accurate process of approximation. In the same manner the calculus, when in its infancy, borrowed methods of integration from geometry, which, though they have since been abandoned for the more useful and accurate methods of approximation, yet merit notice for their elegance, as well as because they constitute the particular connexion with geometry, which first led philosophers to the discovery of the calculus.

The problem which called this science into existence (Geometry, Introduction, p. xxv.), was "to draw a tangent to a given curve," and hence the differential calculus, immediately after its first discovery, was called "*the method of tangents.*" Problems of another kind presented themselves, which proposed the discovery of the curve from some given property of its tangent. As the former depended on what is now called "differentiation," so the latter depended on what is now called "integration."

The integral calculus, when in its infancy, was therefore

called "the inverse method of tangents." As the calculus, however, advanced to a greater state of perfection, and became more extended in its applications, these denominations were necessarily abandoned, being in no respect adequate to the extent of the science. They include no application of either calculus but a geometric one, and even in that, contemplate no differential coefficient beyond the first.

The "inverse method of tangents" consisted in constructing the curve represented by a given differential equation of the first order. If the equation be solved for $\frac{dy}{dx}$, let this be y'. The subtangent is therefore $\frac{y}{y'}$, and the tangent is $\frac{y.\sqrt{1+y'^2}}{y'}$. Hence by means of an equation between the ordinate y and the differential coefficient y', the curve may be constructed by points, and this will represent the integral of the proposed equation.

(397.) Let the proposed differential equation be $\mathrm{F}(xyy')=0$, y' being the first differential coefficient. Let the curve be assumed to pass through a point, of which the co-ordinates are $x=a$ and $y=b$, a and b being values which do not render y' in the equation $\mathrm{F}(xyy')=0$ imaginary. The equation $\mathrm{F}(aby')=0$ will give the value of y', by which the position of the tangent will be known. A point indefinitely near the assumed point, and also upon the tangent, being assumed, and its co-ordinates, in like manner, substituted in the proposed equation, another value of y' may be deduced, which will determine the direction of another tangent. Then a third point being assumed upon this second tangent indefinitely near the second assumed point, a third tangent may be found, and by continuing the process, and not producing the several tangents beyond the several assumed points, a polygon will be determined. The smaller the

distances between the several points are assumed, the more nearly will this polygon approach to a curve, and the curve, which is its *limit*, when the several distances are supposed actually to vanish, is the geometric representation of the integral of the proposed equation.

(398.) A still more accurate and rapid approximation to the curve may be obtained by the following process. Let the equation $F(xyy') = 0$ be differentiated, and the value of the second differential coefficient obtained, as a function of the two variables and the first differential coefficient. Hence may be found the radius of the circle osculating at any proposed point. As before, let a point be assumed, and the tangent at that point found by the proposed equation, and thence the normal. The radius of the osculating circle being determined in the manner already described, let a part equal to it be assumed upon the normal in a direction determined by the sign of the second differential coefficient (151.), and let a small arc of the osculating circle passing through the given point be described. Upon this arc, and near the given point, let another point be assumed, and the circle osculating at that point being found as before, a third point may be assumed upon its arc, and so on.

By this process a polygon will be found, the sides of which are circular arcs, and the smaller these arcs are assumed, so much the nearer will the polygon approach to the curve which represents the integral of the proposed equation. The limit of this polygon, when its sides actually vanish, is the geometric representation of the integral of the proposed equation. The first point, *arbitrarily* assumed in these cases, represents the arbitrary constant.

(399.) If the proposed differential equation be of the second order, it is necessary not only arbitrarily to assume a point through which the curve is supposed to pass, but also the direction of the tangent at that point. This is equivalent to

assigning particular values to x, y, and y', in the equation $F(xyy'y'') = 0$. Hence the value of y'' is determined, and the direction of the curvature and the radius of the osculating circle are known. Proceeding then as in the last case, a polygon, whose sides are small circular arcs, may be determined, the limit of which represents the integral of the sought equation.

(400.) In approximating to the integrals of equations of the higher orders, the osculating parabolas (134.) are used, their several parameters representing the arbitrary constants. The osculating parabola of the second order may also supply the place of the osculating circle in the former cases.

(401.) When the variables in the proposed differential equation are separable, its integral may be otherwise represented by geometrical construction. Let it be reduced to the form

$$\text{Y}dy + \text{X}dx = 0,$$

where Y is a known function of y, and X of x.

Let two curves be constructed relatively to the same axes of co-ordinates, represented by the equations

$$y = \text{X},$$
$$x = \text{Y},$$
$$\therefore \int(\text{X}dx + \text{Y}dy) = \int(ydx) + \int(xdy) = 0.$$

But the area of any part of the first curve intercepted between the axis of y, and any proposed value of y, represents the first integral; and the area of the second intercepted between the axis of x, and the value of x corresponding to the same value of y, represents the other. Their combination, therefore, represents the integral of the sought equation.

The preceding results also show that every differential equation between two variables has an integral, a theorem which was before established in Sect. XVI.

SECTION XXVIII.

The problem of trajectories and other geometrical applications of the integral calculus.

(402.) Amongst the different questions to which the invention of the calculus gave rise, and which were proposed very soon after its invention, one of the most interesting is the "problem of trajectories." In the correspondence between Bernoulli and Leibnitz, on subjects arising out of the new calculus, Bernoulli proposed the solution of the problem, "to find the curve which intersects at right angles a system of curves of the same kind described according to some given law."

This problem, he considered, would lead to the solution of the physical problem, to determine the path of a ray of light through the atmosphere, light being supposed to be propagated according to the Huygenian hypothesis. The problem soon became generalised to that of the determination of the curve which intersects a system of similar curves at the same angle; such a curve is called a trajectory * of the proposed system of curves, and if it intersect them at right angles, it is called the rectangular trajectory.

By "similar curves," is here meant curves whose equations having the same form, differ only in the value of one of the constants, which we shall call in general the variable parameter.

* The term "trajectory," used here, has no relation to the same term used in physics, where it signifies an orbit described by a projectile round a centre of force.

(403.) Let the equation of the proposed system of curves be $F(xyc) = 0$, the constant c representing the variable parameter, and for every particular value of which the equation $F(xyc) = 0$ represents some one of the proposed system of curves. Let the equation of the sought trajectory be $f(xy) = 0$. Let the differential coefficient deduced from the equation $F(xyc) = 0$ be p, the variable constant c being eliminated, and the value of p obtained as a function of the variables x, y, and the other constants. This value of p being independent of c, will be common to the entire of the proposed system of curves. Let the differential coefficient deduced from the equation $f(xy) = 0$ be $\frac{dy}{dx}$, and let the angle at which the sought trajectory is to intersect the proposed system of curves be φ. Hence

$$\tan.\varphi = \frac{\frac{dy}{dx} - p}{1 + p\frac{dy}{dx}},$$

$$\therefore \frac{dy}{dx}(p \tan.\varphi - 1) + p + \tan.\varphi = 0.$$

This being integrated, considering $\tan.\varphi$ as a constant quantity, will give the equation $f(xy) = 0$ of the sought trajectory. It is obvious that p expresses a given function of the variables x, y.

(404.) In order to find the rectangular trajectory, let the above equation be divided by $\tan.\varphi$, by which it becomes

$$\frac{dy}{dx}(p - \cot.\varphi) + p \cot.\varphi + 1 = 0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad [1].$$

If $\varphi = 90^{\circ}$, $\therefore \cot.\varphi = 0$, $\therefore$ the equation becomes

$$pdy + dx = 0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad [2],$$

the integral of which is the rectangular trajectory.

(405.) If the variable parameter c be not eliminated by the given equation $F(xyc) = 0$ and its immediate differential,

then p in [1] will be a function of c, as well as of the variables. In this case the differential equation of the trajectory may be found by eliminating c by the equation [1] and the equation of the proposed system of curves, or by [1] and the differential of that equation. As an arbitrary constant is always introduced, there is always a system of trajectories.

We shall now subjoin a few geometrical problems illustrative of these principles.

PROP. CIV.

(406.) *A system of parabolas having a common vertex and axis, or hyperbolas having a common centre and asymptotes, is given, to find the trajectory intersecting them at a given angle.*

The equation of parabolic and hyperbolic curves in general is

$$y = cx^m,$$

$$\therefore p = mcx^{m-1} = m\frac{y}{x}.$$

Hence the differential equation of the sought trajectory is

$$dy(m\frac{y}{x} - \cot.\varphi) + m\frac{y}{x}\cot.\varphi\, dx + dx = 0,$$

$$\therefore mydy + xdx + \cot.\varphi(mydx - xdy) = 0.$$

This equation being homogeneous, and of the first order, is integrated by (313.). We shall not pursue the general integral here, as its results have not any particular interest.

If $m = 1$, the curve is the right line. In this case the equation becomes

$$(y - x\cot.\varphi)dy + (x + y\cot.\varphi)dx = 0,$$

or

$$xdx + ydy + \cot.\varphi(ydx - xdy) = 0.$$

This becomes integrable by dividing it by $x^2 + y^2$; and since

$$\int\left(\frac{xdx+ydy}{x^2+y^2}\right)=l\sqrt{x^2+y^2},$$

$$\int\frac{ydx-xdy}{x^2+y^2} = \tan.^{-1}\frac{x}{y}.$$

Therefore the equation of the trajectory is

$$l(\sqrt{x^2+y^2}) + \cot.\phi \tan.^{-1}\frac{x}{y} = c,$$

c being an arbitrary constant.

Let $z^2 = x^2 + y^2$, and $\omega = \tan.^{-1}\frac{x}{y}$, ∵

$$lz + \omega \cot.\phi = c;$$

or, if when $\omega = 0$, we suppose that $z = 0$, ∵ $c = 0$, and the equation assumes the form

$$z = e^{-\omega . \cot.\phi}.$$

Let $e^{-\cot.\phi} = a$, ∵

$$z = a^{\omega},$$

which is the logarithmic spiral, of which it is a characteristic property to intersect at the same angle all lines through its pole *.

In this case, for the rectangular trajectory $\cot.\phi = 0$, ∵ z is constant, which shows that the solution is a circle, whose centre is at the origin and radius arbitrary.

In general, the rectangular trajectory of the system of parabolas is determined by integrating

$$mydy + xdx = 0,$$

$$\therefore my^2 + x^2 = c.$$

If $m > 0$, the trajectories are a system of similar ellipses,

* Geom. (433.).

having a common centre at the common vertex of the system of parabolas, and an axis coincident with the axis of the system, the ratio of their axes being $1 : \sqrt{m}$. If the parabola be the parabola of the second degree, $m = 2$. This case is remarkable for having been the first to which the problem of trajectories was applied. The general problem having been proposed by Bernoulli, Leibnitz gave a general method of solving it, and effected the solution in this instance as an example. Leibnitz's method was founded upon the variation of the constant c in passing from one curve of the proposed system to another, from which he deduced his method of differentiation *de curvâ in curvam.*

If $m < 0$, the proposed equation represents a system of hyperbolas having a common centre and asymptotes, and the trajectories are also a system of conical hyperbolas, of which the axes coincide with the common asymptotes of the system.

If the given system of hyperbolas be equilateral, the trajectories are also equilateral hyperbolas.

PROP. CV.

(407.) *To determine the trajectory of a system of circles touching a given right line at a given point.*

The right line being assumed as axis of y, and the given point as origin, the equation of the circles is

$$y^2 + x^2 - 2rx = 0,$$

$$\therefore py + (x - r) = 0,$$

$$\therefore r = py + x.$$

This being substituted in the first, we find

$$p = \frac{y^2 - x^2}{2yx}.$$

Hence the differential equation of the trajectory is

$$dy(y^2-x^2-2\cot.\phi\cdot yx)+[2yx+\cot.\phi(y^2-x^2)]dx=0.$$

This equation being homogeneous, may be integrated by (313.).

If the rectangular trajectories be sought, $\cot.\phi = 0$, $\because$

$$dy(y^2 - x^2) + 2yxdx = 0.$$

This is immediately integrable by putting $z = x^2$, by which the equation becomes

$$dy = -\frac{(ydz-zdy)}{y^2},$$

$$\because y = -\frac{z}{y} + c,$$

$$\because y^2 = -z + cy,$$

$$\because y^2 + x^2 - cy = 0.$$

Hence the system of rectangular trajectories are circles passing through the given point of contact, and having their centres upon the given tangent.

(408.) Instances of the class of problems which gave the name of the inverse method of tangents to the integral calculus are not infrequent. In these, some property of the tangent, or some line depending on the tangent, as the normal, subtangent, subnormal, &c. is given, to determine the curve. It will be sufficient here to give a few examples of these, to show that they are always capable of solution by the integration of equations of two variables.

PROP. CVI.

(409.) *To determine the curve in which the normal is a given function of the intercept of the axis of* x *between it and the origin.*

The intercept between the normal and the origin is the sum of the subnormal and the value of x for the point

where the normal meets the curve. Hence the problem is reduced to the integration of the equation

$$y\sqrt{1+\frac{dy^2}{dx^2}} = \mathrm{F}\left(x+\frac{ydy}{dx}\right).$$

The integration of this equation will solve the proposed question.

If the normal be supposed equal to the intercept, the equation becomes

$$y\sqrt{1+\frac{dy^2}{dx^2}} = x + y\cdot\frac{dy}{dx},$$

$$\therefore y^2\left(1+\frac{dy^2}{dx^2}\right) = x^2 + y^2\cdot\frac{dy^2}{dx^2} + 2yx\cdot\frac{dy}{dx},$$

$$\therefore \frac{dy}{dx} = \frac{y^2-x^2}{2yx} = \frac{1}{2}\left(\frac{y}{x}-\frac{x}{y}\right).$$

The integral of which is

$$y^2 + x^2 - 2rx = 0,$$

r being an arbitrary constant. The curve sought is therefore the circle.

If the normal be the ordinate of a parabola, of which the abscissa is the intercept

$$\mathrm{F}\left(x + y\frac{dy}{dx}\right) = 2a\left(x + y\frac{dy}{dx}\right),$$

$$\therefore y^2\left(1+\frac{dy^2}{dx^2}\right) = 2a\left(x+\frac{ydy}{dx}\right).$$

Obtaining from this the value of $y\frac{dy}{dx}$, and dividing by the radical, we find

$$\frac{a - y\dfrac{dy}{dx}}{\sqrt{a^2+2ax-y^2}} + 1 = 0.$$

Integrating this, and supplying the constant, we find an equation of the form

$$y^2 + x^2 - 2rx + \mathrm{A} = 0,$$

which is that of a circle. This is the general solution.

The singular solution is obtained by putting the radical $= 0$ (339.), $\because$ it is

$$y^2 = 2ax + a^2,$$

which is the equation of a parabola. This parabola is the curve to which all the circles included in the general solution are tangents.

If

$$\text{F}\left(x + y\frac{dy}{dx}\right) = x + a,$$

a being a constant quantity, we have

$$y\frac{dy}{dx} = a,$$

$$\because y^2 = 2ax + c,$$

which shows that the parabola is the only curve whose subnormal is constant.

(410.) Geometrical questions which relate to the osculating circle are solved by the integration of differential equations of the second order. The following proposition furnishes an example of this.

PROP. CVII.

(411.) *To determine the curve in which the radius of the osculating circle is a given function of the normal.*

This problem, reduced to an equation, is

$$-\frac{(dy^2+dx^2)^{\frac{3}{2}}}{d^2ydx} = \text{F}\left\{y\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\},$$

the integration of which will solve the problem.

If the radius be equal to the normal, the equation becomes

$$\frac{dy^2+dx^2}{d^2ydx} = -\frac{y}{dx},$$

$$\therefore dy^2 + dx^2 + yd^2y = 0.$$

The first integral of which is

$$ydy + xdx = cdx,$$

which being again integrated, gives

$$y^2 + x^2 - 2cx + c' = 0,$$

which is the equation of a circle of which the centre is on the axis of x.

(412.) The student will find no difficulty in reducing geometrical questions relating to contact or curvature to equations. These equations are generally of the first or second degree, and integrable by the rules already established. To extend the examples on this farther would occupy more space here than the difficulty of the investigation requires. We shall therefore conclude this section with the following proposition, as an example of another and different species of problem.

PROP. CVIII.

(413.) *A system of parabolas having a common vertex and axis, or hyperbolas having common asymptotes, being given, to find the curve which intersects them all, so that the areas included by the co-ordinates of the point of intersection, and the arc of the parabola or hyperbola between that point and the axis of* y, *shall be constant.*

Let the equation of the proposed system of curves be

$$y = px^m.$$

The area included by the co-ordinates and the arc is

$$\int ydx = p\int x^m dx = \frac{px^{m+1}}{m+1}.$$

No constant is added, as the area is supposed to commence when $x = 0$.

If $m = -1$, the integral is

$$\int y dx = pl(x),$$

and if $m < -1$, the area is infinite when $x = 0$, which is also the case when $m = -1$. These cases will then be excepted in the following investigation, which will therefore only apply to parabolas, and to such hyperbolas as have $m > -1$.

Let the given area be A, ∴

$$A = p\frac{x^{m+1}}{m+1}.$$

Eliminate p by this and the equation of the proposed system, and the result is

$$yx = A(m+1),$$

which is the equation of a common hyperbola.

SECTION XXIX.

Of the integration of total differential equations of the first degree of several variables, which satisfy the conditions of integrability.

(414.) A total differential equation of the first order between three variables must always come under the formula

$$Pdx + Qdy + Rdz = 0.$$

If the first member of this equation satisfy the criterion of integrability, (286.), for functions of three variables, its integral may be immediately obtained by the rules for these functions, and will be of the form

$$F(xyz) + C = 0,$$

C being an arbitrary constant.

If any one of the three variables be capable of being separated from the other two, the equation may be integrated by the rules for the integration of functions of two variables. For if z be separable from x and y, the equation may be reduced to the form

$$\text{z}dz = \text{P}dx + \text{Q}dy,$$

where Z represents a function of z.

This separation is easily effected whenever the given equation has the form

$$\text{Z}(\text{P}dx + \text{Q}dy) + \text{R}\text{Z}'dz = 0,$$

by dividing the whole equation by ZR; P, Q, and R, being functions of x and y only.

(415.) If the proposed equation be not an exact differential of an equation of three variables, it may sometimes be rendered so by the introduction of a factor. To determine the condition under which it is rendered integrable by a multiplier, let

$$\text{TP}dx + \text{TQ}dy + \text{TR}dz = 0$$

be the equation after the introduction of the factor T. If this be an immediate or exact differential, it follows that

$$\text{TP}dx + \text{TQ}dy = 0$$

is the exact differential of the same equation, z being considered constant, and, in like manner, that

$$\text{TP}dx + \text{TR}dz = 0,$$

$$\text{TQ}dy + \text{TR}dz = 0,$$

are the immediate differentials, y and x being taken successively constant. It appears, therefore, that any factor which renders the total equation integrable, also renders all the partial equations integrable; and it is obvious, that if the same factor render all three partial differential equations integrable, it will render the total equation also integrable. In order that the three partial equations should be exact differentials, it is necessary, (286.), that the conditions

$$\frac{d(\text{TP})}{dy}=\frac{d(\text{TQ})}{dx},\quad \frac{d(\text{TP})}{dz}=\frac{d(\text{TR})}{dx},\quad \frac{d(\text{TQ})}{dz}=\frac{d(\text{TR})}{dy}$$

should be fulfilled. Hence we find

$$\left.\begin{array}{l} \text{T}\left(\frac{d\text{P}}{dy}-\frac{d\text{Q}}{dx}\right)+\text{P}\frac{d\text{T}}{dy}-\text{Q}\frac{d\text{T}}{dx}=0 \\ \text{T}\left(\frac{d\text{R}}{dx}-\frac{d\text{P}}{dz}\right)+\text{R}\frac{d\text{T}}{dx}-\text{P}\frac{d\text{T}}{dz}=0 \\ \text{T}\left(\frac{d\text{Q}}{dz}-\frac{d\text{R}}{dy}\right)+\text{Q}\frac{d\text{T}}{dz}-\text{R}\frac{d\text{T}}{dy}=0 \end{array}\right\}\cdots\cdots[1].$$

Multiplying the first by R, the second by Q, and the third by P, adding them, and dividing the result by its factor T, we obtain

$$\text{R}\left(\frac{d\text{P}}{dy}-\frac{d\text{Q}}{dx}\right)+\text{Q}\left(\frac{d\text{R}}{dx}-\frac{d\text{P}}{dz}\right)+\text{P}\left(\frac{d\text{Q}}{dz}-\frac{d\text{R}}{dy}\right)=0\cdots\cdot[2].$$

This equation must therefore be satisfied by the proposed equation when it is capable of being rendered integrable by a multiplier. On the other hand, if the proposed equation do not satisfy this condition, there is no multiplier by which it can be rendered integrable. It will not be difficult to generalise these principles, and obtain conditions of integrability for equations of four or more variables. The number of equations of condition is, however, greater, being always the number of combinations of two, which can be made with $m-1$ things, m being the entire number of variables. The number of equations of condition is, therefore, in general, $\frac{m-1.m-2}{1.2}$.

Equations of three or more variables, therefore, differ from equations of two in the same manner as functions of two or more variables differ from functions of a single variable. Equations of two variables, and functions of one, can always be integrated, either exactly or by approximation; but there are cases in which differential equations of

three or more variables, and functions of two or more, admit of no integral either exact or by approximation.

(416.) When the condition [2] is fulfilled, the integration of the proposed differential equation of three variables may be shown to depend upon the integration of an equation of two variables. Let z be supposed constant, so that $dz = 0$, and the proposed equation becomes

$$\mathrm{P}dx + \mathrm{Q}dy = 0.$$

Let this be integrated, and the result will have the form

$$\mathrm{U} + \mathrm{Z} = 0,$$

where U is a function of x, y, z, and Z an arbitrary function of z, which takes the place of the arbitrary constant.

Let this equation be differentiated with respect to x, y, and z, and let the function Z be so assumed, as to render the differential equation thus deduced identical with the proposed equation. The value of the function which satisfies this condition being substituted for it, gives the sought integral.

The following examples of the application of this rule will render it more easily apprehended.

Ex. 1. Let the proposed equation be

$$(y + z)dx + (x + z)dy + (x + y)dz = 0.$$

Let $dz = 0$, ∴

$$(y + z)dx + (x + z)dy = 0,$$

$$\therefore \frac{dx}{x+z} + \frac{dy}{y+z} = 0.$$

Since z is considered constant, the integral of this is

$$(x + z)(y + z) + \mathrm{Z} = 0.$$

To determine the function Z, which will render this the integral sought, let it be differentiated with respect to x, y, z, and the result is

$$(y + z)dx + (x + z)dy + (y + x + 2z)dz + d\mathrm{Z} = 0.$$

That this may be identical with the proposed equation, we must have

$$2zdz + dz = 0,$$
$$\therefore z^2 + z = c.$$

Hence the integral sought is

$$xy + zy + zx + c = 0.$$

Ex. 2. Let the proposed equation be

$$zdx + xdy + ydz = 0.$$

In this case $P = z$, $Q = x$, and $R = y$, by which it appears that the equation [2] is not fulfilled, and therefore the proposed equation is not integrable. If this equation were submitted to the preceding process, we should find that z could not be disengaged from x and y, so that we should find $Z = F(xyz)$.

Ex. 3. Let the proposed equation be such, that

$$P = y^2 + yz + z^2,$$
$$Q = x^2 + xz + z^2,$$
$$R = x^2 + xy + y^2.$$

In this case the criterion [2] is satisfied. If $dz = 0$, we have

$$\frac{dx}{x^2+xz+z^2} + \frac{dy}{y^2+yz+z^2} = 0.$$

Since z is constant, the integral of this is

$$\frac{2}{z\sqrt{3}}\left\{\tan.^{-1}\frac{z+2x}{z\sqrt{3}} + \tan.^{-1}\frac{z+2y}{z\sqrt{3}}\right\} = f(z).$$

If an arc be a function of z, its tangent must be also a function of z; and hence by taking the tangents of both sides, we find

$$\frac{(x+y+z)z}{z^2-zx-zy-2xy} = Z.$$

Differentiating this, and identifying the result with the proposed equation, we find

$$2(x^2z + 3xyz + y^2z + z^2x + z^2y + x^2y + y^2x)dz$$
$$+ (z^2 - zx - zy - 2xy)^2dz = 0.$$

Eliminating the latter parenthesis by

$$z^2 - zx - zy - 2xy = \frac{(x+y+z)z}{\mathrm{z}},$$

and expunging the common factor $(x+y+z)$, we obtain

$$2(xy+yz+xz)\mathrm{z}^2dz + (x+y+z)z^2\,d\mathrm{z} = 0.$$

Also by the integral just obtained, we find

$$xz + yz = \frac{z^2\mathrm{z} - z^3 - 2xy\mathrm{z}}{\mathrm{z}+1}.$$

Making this substitution, and dividing by the common factor $2z(z^2 - xy)$, we obtain

$$\mathrm{z}(\mathrm{z}-1)dz + zd\mathrm{z} = 0,$$

$$\therefore \frac{dz}{z} = \frac{d\mathrm{z}}{\mathrm{z}} - \frac{d\mathrm{z}}{\mathrm{z}-1}, \quad \therefore z = \frac{c\mathrm{z}}{\mathrm{z}-1},$$

$$\therefore \mathrm{z} = \frac{z}{z-c}.$$

Hence the integral required is

$$z\,(xy + xz + yz) - c(x + y + z) = 0.$$

(417.) If the proposed differential equation exceed the first degree, these methods are only applicable when it can be decomposed into rational factors of the form

$$\mathrm{P}dx + \mathrm{Q}dy + \mathrm{R}dz = 0;$$

this being the only form it can have when it is an immediate differential.

If for example the proposed equation be

$$\mathrm{P}dx^2 + \mathrm{Q}dy^2 + \mathrm{R}dz^2 + 2\mathrm{S}dxdy + 2\mathrm{T}dxdz + 2\mathrm{V}dydz = 0.$$

When this is solved for dz, the quantity under the radical is

$$(\mathrm{T}^2 - \mathrm{PR})dx^2 + 2(\mathrm{TV} - \mathrm{RS})dxdy + (\mathrm{V}^2 - \mathrm{QR})dy^2.$$

It is necessary that this should be a complete square, which can only take place under the condition

$$(\mathrm{TV} - \mathrm{RS})^2 - (\mathrm{T}^2 - \mathrm{PR})(\mathrm{V}^2 - \mathrm{QR}) = 0.$$

SECTION XXX.

Integration of total differential equations which do not satisfy the criterion of integrability.

(418.) Differential equations, which do not satisfy the criterion [2] established in the last section, were long considered as absurd or impossible relations; and all questions, whose solution was reduced to such equations, were considered as involving some contradiction, as is the case when the solution involves the even roots of negative quantities.

MONGE, however, has shown that this is not the case, and that such equations indicate a real relation between the variables. It happens, however, that the integral of such an equation is not, like those which satisfy the criterion, one equation between three variables, but it is expressed by two equations between three variables which must subsist together, and which involve an arbitrary function of one of the variables.

(419.) The integral of an ordinary differential equation of three variables, which satisfies the criterion of integrability, would, if represented geometrically, be a curved surface, since the integral is an equation of three variables. The integral of an equation which does not satisfy the criterion, if represented geometrically, would be a class of curves of double curvature, enjoying some common characteristic property.

For each value of the arbitrary function which enters the system of equations, there is a particular curve of double curvature. The part of the equations which does not depend on this function, being common to all particular

values of the function, gives the general geometric character to the class of curves.

(420.) To determine the system of equations which represents the integral of any given equation of this kind, let z be considered as constant, ∴

$$\mathrm{P}dx + \mathrm{Q}dy = 0.$$

Let T be the factor which renders this integrable, and let $\mathrm{U} + \mathrm{Z} = 0$ be the integral of

$$\mathrm{TP}dx + \mathrm{TQ}dy = 0.$$

Differentiating $\mathrm{U} + \mathrm{Z} = 0$, and identifying it with

$$\mathrm{TP}dx + \mathrm{TQ}dy + \mathrm{TR}dz = 0,$$

we obtain

$$\mathrm{U} + \mathrm{Z} = 0,$$

$$\frac{d\mathrm{Z}}{dz} = \mathrm{TR} - \frac{d\mathrm{U}}{dz}.$$

In this case $\frac{d\mathrm{Z}}{dz}$ is not a function of the variable z alone, for if it were, the equation would be integrable by the process (416.), and would fulfil the criterion, which is contrary to hypothesis. These equations must then subsist together, Z being an arbitrary function of z. Let $\mathrm{Z} = \mathrm{F}(z)$, and $\frac{d\mathrm{Z}}{dz} = \mathrm{F}'(z)$, ∴

$$\mathrm{U} + \mathrm{F}(z) = 0,$$

$$\frac{d\mathrm{U}}{dz} + \mathrm{F}'(z) - \mathrm{TR} = 0,$$

which are two equations between the three variables, and taken together, represent a relation between xyz, which satisfies the proposed differential equation.

(421.) Since the function $\mathrm{F}(z)$ is absolutely arbitrary, it follows that there are an infinite number of systems of two equations which satisfy the proposed equation, and that, therefore, it has an infinite number of systems of integrals. If the integral be represented geometrically for each form

assigned to the function $F(z)$, there is a different curve of double curvature. The terms U, T, and R, however, not changing with the form of this function, will give some common character to all these curves.

As an example, let the proposed equation be

$$\frac{dz}{z-c}=\frac{xdx+ydy}{x(x-a)+y(y-b)}.$$

In this case,

$$Pdx + Qdy = \frac{xdx+ydy}{x(x-a)+y(y-b)},$$

$$R = -\frac{1}{z-c}.$$

Let

$$T = x(x-a) + y(y-b).$$

We find

$$U = x^2 + y^2.$$

Hence

$$x^2 + y^2 = F(z),$$

$$x(x-a) + y(y-b) = F'(z) \cdot (z-c):$$

in which $F(z)$ is absolutely arbitrary.

SECTION XXXI.

Of the integration of partial differential equations of the first order.

(422.) The integration of partial differential equations is a part of the calculus which has not yet reached that state of perfection which might enable an elementary author to introduce such an exposition of its principles as is suitable to the class of students for whose use his work is intended. In this, as in some other parts of the calculus, the utmost which

can be attempted in the present work is to explain the methods of integrating some particular classes of equations, which are most suited to our object, referring students, desirous of further information, to such works as the complete treatise of Lacroix.

(423.) The most simple class of partial differential equations are those which involve but one partial differential coefficient. The integration of these may be always reduced either to the integration of functions of one variable, or to the integration of equations of two variables. Let p be a partial differential coefficient of z with respect to x, *i. e.* $p = \frac{dz}{dx}$, and let u be a function of x and several other variables, and let the given partial differential equation be

$$\mathrm{F}(p, u) = 0.$$

First, suppose that u does not include the variable z. Let the equation in this case be solved for p, and its value substituted, ∴

$$\frac{dz}{dx} = f(u),$$

$$\therefore \; dz = f(u) \cdot dx.$$

Since the coefficient $\frac{dz}{dx}$ was obtained by differentiating z as a function of x only, all the other variables being considered constant, so the integration must be effected upon the same supposition. Let x therefore be considered to be the only variable in u, all the others being taken as constants, and let the integral of $f(u) \cdot dx$ be found by the rules for the integration of functions of one variable. Let the integral be

$$z = \mathrm{U} + \mathrm{C},$$

U expressing the function of all the variables obtained by the integration, and C the arbitrary constant.

Since all functions of the variables, not including x, which

entered the original function, necessarily disappeared by the differentiation which gave $\frac{dz}{dx}$, it therefore follows that an arbitrary function of these variables should be introduced in the integration. We must then consider C, not as an arbitrary constant, but an arbitrary function of all the variables, except z and x.

(424.) If, however, the function u contain z as well as x, the equation may be integrated as a differential equation between z and x, the other variables being considered as constants; and in place of an arbitrary constant, introducing in the integration an arbitrary function of the other variables.

(425.) The most general partial differential equation of the first degree, including two partial differential coefficients, is of the form

$$\text{P}p + \text{Q}q = \text{V} \cdot \cdot \cdot \cdot [1].$$

We shall consider p and q as the partial differential coefficients of z with respect to the variations of x and y, $\therefore$

$$p = \frac{dz}{dx}, \quad q = \frac{dz}{dy}.$$

If other variables enter the functions P, Q, V, besides x, y, and z, they are to be treated as constants; and in place of arbitrary constants, arbitrary functions of these other variables should be introduced in the integral. In what follows, we shall consider the equation [1] to include only the variables x, y, and z.

(426.) By the definitions of partial differentials (94.), we have

$$dz = pdx + qdy \cdot \cdot \cdot \cdot [2].$$

Eliminating p by this equation and [1], the result will be

$$\text{P}dz - \text{V}dx = q(\text{P}dy - \text{Q}dx) \cdot \cdot \cdot \cdot [3].$$

This equation must be satisfied independently of q, since in

the proposed equation q is indeterminate. The integration proposed may be reduced to two cases:

1°. When $\mathrm{P}dz - \mathrm{V}dx$ does not contain y, nor $\mathrm{P}dy - \mathrm{Q}dx$, z, or what amounts to the same, where these variables may be disengaged from them.

2°. Where one or both of these quantities contain all three variables x, y, z.

(427.) 1°. If the quantity

$$\mathrm{P}dz - \mathrm{V}dx$$

do not contain y, it either is an exact differential of a function of x, z, or may be rendered so by a factor. Let the factor which renders it exact be μ, and let the function of which it is a differential be M, $\therefore$

$$\mathrm{P}dz - \mathrm{V}dx = \frac{d\mathrm{M}}{\mu}.$$

In like manner, since

$$\mathrm{P}dy - \mathrm{Q}dx$$

does not contain z, we have

$$\mathrm{P}dy - \mathrm{Q}dx = \frac{d\mathrm{M}'}{\mu'}.$$

Hence the equation [3] becomes

$$d\mathrm{M} = \frac{q\mu}{\mu'}d\mathrm{M}'.$$

This is only integrable when $\frac{q\mu}{\mu'}$ is a function of M'. Let

$\mathrm{F}'(\mathrm{M}') = \frac{q\mu}{\mu'}d\mathrm{M}'$, and let the integral of this be $\mathrm{F}(\mathrm{M}')$, $\therefore$

$$\mathrm{M} = \mathrm{F}(\mathrm{M}'),$$

where $\mathrm{F}(\mathrm{M}')$ is an arbitrary function of M'.

Had q been eliminated by [1] and [2] instead of p, the result would have been

$$\mathrm{Q}dz - \mathrm{V}dy = p(\mathrm{Q}dx - \mathrm{P}dy),$$

and the integration would, in this case, depend on the integration of the formulæ

$$Q dz - V dy,$$
$$Q dx - P dy.$$

It therefore follows in general, that if any two of the three equations

$$\left.\begin{array}{l} P dy - Q dx = 0 \\ P dz - V dx = 0 \\ Q dz - V dy = 0 \end{array}\right\} \cdots\cdots [4],$$

be integrated, and that their integrals be M, M', each of which represent functions of x, y, z, the integral of [1] will be

$$M = F(M'),$$

the form of the function being absolutely arbitrary.

(428.) We have supposed that each of these formulæ [4] excludes one of the variables. The principles we have just established are, however, applicable, even if any two of the formulæ [4] included all the three variables, provided that the third contained only the two variables whose differentials are engaged in it. For this being integrated as a function of two variables, and its integral being $M = 0$, either of the two variables may be eliminated by means of this integral; and either of the other two formulæ, including the three variables, by which a formula may be obtained, including only two of the three variables and their differentials, whose integral M' being obtained, the integral of [1] will be $M = F(M')$, the function as before being arbitrary.

(429.) Even if the three equations [4] should all contain the three variables, yet, if any two of them, and therefore the third (since they are not independent), be satisfied by the equations $M = 0$ and $M' = 0$, the integral of [1] will be $M = F(M')$ as before. To prove this, it will be necessary to show that the differential of $M = F(M')$ satisfies the conditions [4] independently of the form of the function.

Let the differential of the equation $M = F(M')$ be

$$dM = F'(M')dM'.$$

That this may be satisfied independently of $F'(M')$, the form of which depends on that of $F(M')$, it is necessary that the conditions

$$dM = 0, \quad dM' = 0,$$

should be fulfilled. Since M and M' are functions of x, y, z, their differentials must be of the form

$$A dx + B dy + C dz = 0,$$
$$A' dx + B' dy + C' dz = 0.$$

If the equation $M = F(M')$ be differentiated with respect to z and x, we shall have

$$A dx + C dz = F'(M')(A' dx + C' dz);$$

and if it be differentiated with respect to z and y,

$$B dy + C dz = F'(M')(B' dy + C' dz).$$

Substituting for $\dfrac{dz}{dx}$ and $\dfrac{dz}{dy}$ their values p and q, we find

$$[C - C'F'(M')]p + A - A'F'(M') = 0,$$
$$[C - C'F'(M')]q + B - B'F'(M') = 0.$$

Deducing hence the values of p and q, and substituting them in [1], we find

$$AP + BQ + CV = F'(M')(A'P + B'Q + C'V).$$

Substituting in the values of dM, dM', the values of dx, dy, obtained from [4], and taking out the common factor dz, we find

$$\left.\begin{aligned} AP + BQ + CV &= 0 \\ A'P + B'Q + C'V &= 0 \end{aligned}\right\} \quad \ldots\ldots \quad [5].$$

Hence the above equation is satisfied independently of the form of $F'(M')$. It follows, therefore, that unless the differentials of M and M' combined with [4] satisfy the conditions [5], the equation $M = F(M')$ cannot be the integral of the proposed equation.

It is obvious that $M = a$ and $M' = b$ are particular integrals, a and b being arbitrary constants, for $F(M')$ may be

considered constant, and $\therefore$ the equation $M = F(M')$ becomes, in this particular case, $M = 0$; or $F^{-1}(M)$ * may be constant; in which case the equation becomes $M' = b$.

(430.) If $V = 0$, the equation [1] becomes

$$Pp + Qq = 0,$$

and the equations [4] become

$$Pdy - Qdx = 0,$$
$$dz = 0.$$

Hence $z = M$, and there can be only two variables in the first, the integral of which being M', the complete integral will be $z = F(M')$.

For example, let the proposed equation be $px = qy$, $\therefore$ $xdy - ydx = 0$, $\therefore y = Ax$, and $a = F(z)$, $\therefore y = F(z) \cdot x$, or $z = f\left(\frac{y}{x}\right)$, which is the general equation of conical surfaces.

If $py = qx$, $\therefore P = y$, $Q = -x$, $\therefore$

$$ydy + xdx = 0,$$
$$\therefore y^2 + x^2 = M',$$
$$\therefore z = F(x^2 + y^2),$$

which is the general equation of surfaces of revolution round the axis of z.

Let $q = Pp$, P not containing z. The integral is

$$z = F(M'), \quad M' = \int T(dx + Py),$$

T being the factor which renders the proposed equation integrable.

If two of the equations [3] contain but two of the three variables, the integration presents no difficulty. For example, let the proposed equation be $px + qy = nz$. Hence

$$xdz = nzdx,$$
$$xdy = ydx,$$

* $F^{-1}(M)$ means a quantity u, such, that $F(u) = M$.

$$\therefore \frac{z}{x^n} = \text{M},$$

$$\frac{y}{x} = \text{M}', \quad \therefore z = x^n \cdot \text{F}\left(\frac{y}{x}\right).$$

This result, applicable to homogeneous functions, has been already obtained in (322.).

(431.) Ex. 1. Let the proposed equation be

$$px^2 + qy^2 = z^2,$$

$$\therefore x^2dz = z^2dx, \quad x^2dy = y^2dx,$$

$$\therefore \frac{1}{z} - \frac{1}{x} = \text{M}, \quad \frac{1}{y} - \frac{1}{x} = \text{M}',$$

$$\therefore \frac{1}{z} - \frac{1}{x} = \text{F}\left(\frac{1}{y} - \frac{1}{x}\right),$$

$$\text{or } \frac{x-z}{zx} = \text{F}\left(\frac{x-y}{yx}\right).$$

Ex. 2. Let the proposed equation be $q = \text{X}p + \text{V}$, where X and V are functions of x only. Hence

$$\text{X}dz + \text{V}dx = 0,$$

$$\text{X}dy + dx = 0,$$

$$\therefore z = -\int\frac{\text{V}dx}{\text{X}} + \text{F}\left(y + \int\frac{dx}{\text{X}}\right).$$

Ex. 3. Let the proposed equation be

$$qxy - px^2 = y^2,$$

$$\therefore x^2dz + y^2dx = 0, \quad x^2dy + xydx = 0.$$

In this case, one of the equations [4] includes but two variables. This being integrated, gives $xy = \text{M}'$. Substituting in the first $\frac{\text{M}'}{x}$ for y, it becomes

$$x^4dz + \text{M}'^2dx = 0,$$

which being integrated, gives

$$z = \tfrac{1}{3}\text{M}'^2x^{-3} + \text{M}.$$

Substituting xy for M', and $\text{F}(xy)$ for M, we find, for the integral sought,

$$3zx = y^2 + 3x\text{F}(xy).$$

Let the equation be

$$px + qy = n\sqrt{x^2 + y^2},$$

$$\therefore xdz = n\sqrt{x^2 + y^2}\,dx, \quad xdy - ydx = 0.$$

The latter being integrated, gives

$$y = \text{M}'x,$$

by which y being eliminated, the former becomes

$$dz = n\sqrt{1 + \text{M}'^2}\,dx,$$

$$\therefore z - nx\sqrt{1 + \text{M}'^2} = \text{M},$$

Hence

$$z = n\sqrt{x^2 + y^2} + \text{F}\left(\frac{y}{x}\right).$$

(432.) 2°. In general, when each of the equations [4] contains all the variables, they cannot be integrated separately, because we cannot suppose two of the variables to change, while the third remains constant. Various analytical artifices have been suggested for obtaining the integral in these cases.

By integration by parts, the equation

$$dz = pdx + qdy,$$

may assume any of the three following forms:

$$z = px + \int(qdy - xdp),$$

$$z = qy + \int(pdx - ydq),$$

$$z = px + qy - \int(xdp + ydq).$$

It frequently happens that we can obtain the sought integral by substituting the value of p or q derived from the proposed equation [1] in any of the preceding.

For example, if p be a function of q, so that $p = \text{Q}$, the last of the preceding equations becomes

$$z = \text{Q}x + qy - \int(x\text{Q}' + y)dq,$$

where $\text{Q}' = \dfrac{d\text{Q}}{dq}$. Hence

$$x\text{Q}' + y = \text{F}'(q),$$

$$\therefore z = \text{Q}x + qy - \text{F}(q),$$

where the function F is arbitrary. The integral for par-

ticular forms of the function $F(q)$ may be found from these equations by eliminating q.

(433.) The integration of partial differential equations of the first order is sometimes effected by the following process. Let the given differential equation be

$$F'(xyzpq) = 0.$$

Let this be solved for either of the partial differential coefficients (p), and let the value of p, thus determined, be substituted in

$$dz = pdx + qdy,$$

by which we obtain an equation of the form

$$dz = f(xyzq)dx + qdy.$$

Let θ be such a function of q, as being considered constant, this equation will become an exact differential; and let its integral be

$$u = F'(xyz\theta) = c,$$

c being an arbitrary constant. This equation being differentiated, θ being considered constant, ought to reproduce the differential equation from which it was obtained; and it should also reproduce it, if θ being considered variable in the first member, c were such a function of θ as would fulfil the condition

$$\frac{du}{d\theta}d\theta = \frac{dc}{d\theta} \cdot d\theta.$$

For differentiating, as if θ and c were both constant, we should get

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz = 0;$$

and differentiating, considering θ variable, and c a function of θ, we should obtain

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz + \frac{du}{d\theta}d\theta = \frac{dc}{d\theta}d\theta.$$

In order that this and the former may be identical, we must therefore have

$$\frac{du}{d\theta}.d\theta = \frac{dc}{d\theta}.d\theta.$$

Hence, if by taking θ as constant, the equation becomes an exact differential, we obtain by integration an equation of the form

$$u = \mathrm{F}(xyz\theta) = f(\theta),$$

the function θ being restricted by the condition

$$\frac{du}{d\theta} = \frac{df(\theta)}{d\theta}.$$

These two equations will satisfy the proposed equation, the function $f(\theta)$ being arbitrary. If this function be determined, the elimination of θ by the two equations will give the integral of the proposed equation.

As an example of the application of this method, let the proposed equation be $z = pq$. Substituting in

$$z = pdx + qdy,$$

the value of p derived from the proposed equation, we find

$$dz = \frac{z}{q}dx + qdy,$$

$$\therefore dy = \frac{qdz - zdx}{q^2}.$$

This will be an exact differential, if $\theta = q - x$ and θ be considered as constant, $\therefore$

$$dy = \frac{(\theta + x)dz - zdx}{(\theta + x)^2},$$

$$\therefore y = \frac{z}{x+\theta} + f(\theta),$$

$$\therefore \frac{z}{(x+\theta)^2} = \frac{df(\theta)}{d\theta}.$$

These two equations conjointly represent the integral of $z = pq$.

(434.) The integration of partial differential equations of the first order is often effected by the introduction of an indeterminate quantity. Let the proposed equation be

$f(px) = \text{F}(qy)$. Let $f(px) = \omega$, $\because \text{F}(qy) = \omega$. Deducing from these the values of p and q, we obtain equations of the forms

$$p = f'(x\omega), \quad q = \text{F}'(y\omega),$$
$$\because dz = f'(x\omega) \cdot dx + \text{F}'(y\omega)dy.$$

Let the integral of $f'(x\omega)dx$, integrated with respect to x be P, and that of $\text{F}'(y\omega)dy$, integrated with respect to y be Q. Hence, considering that P and Q must be also functions of the indeterminate ω, we have

$$f'(x, \omega)dx = \frac{d\text{P}}{dx}dx = d\text{P} - \frac{d\text{P}}{d\omega}d\omega,$$

$$\text{F}'(y, \omega)dy = \frac{d\text{Q}}{dy}dy = d\text{Q} - \frac{d\text{Q}}{d\omega}d\omega.$$

Hence we find

$$dz = d\text{P} + d\text{Q} - \left(\frac{d\text{P}}{d\omega} + \frac{d\text{Q}}{d\omega}\right)d\omega.$$

This equation can only be an exact differential when the quantity within the parentheses is a function of ω, *i. e.*

$$\frac{d\text{P}}{d\omega} + \frac{d\text{Q}}{d\omega} = \varphi'(\omega),$$

$$\because \int\left(\frac{d\text{P}}{d\omega} + \frac{d\text{Q}}{d\omega}\right)d\omega = \varphi(\omega).$$

Hence the combination of equations

$$z + \varphi(\omega) = \text{P} + \text{Q},$$

$$\varphi'(\omega) = \frac{d\text{P}}{d\omega} + \frac{d\text{Q}}{d\omega},$$

where $\varphi(\omega)$ is an arbitrary function, represent the integral of the proposed equation.

As an example of this process, let the proposed equation be $a^2pq = x^2y^2$. Hence

$$\frac{ap}{x^2} = \frac{y^2}{aq},$$

$$\because \frac{ap}{x^2} = \omega, \quad \frac{y^2}{aq} = \omega,$$

$$\therefore f'(x, \omega) = \frac{x^2\omega}{a}, \quad \mathrm{F}'(y, \omega) = \frac{y^2}{a\omega},$$

$$\therefore \mathrm{P} = \frac{x^3\omega}{3a}, \quad \mathrm{Q} = \frac{y^3}{3a\omega}.$$

Hence the integral is represented by the equations

$$3az + 3a\varphi(\omega) = x^3\omega + \frac{y^3}{\omega},$$

$$\varphi'(\omega) = x^3 - \frac{y^3}{\omega^2}.$$

(435.) If the proposed equation of the form

$$\mathrm{P}p + \mathrm{Q}q = \mathrm{V}$$

be homogeneous with respect to the three variables, let $x = tz$ and $y = uz$. The quantities P, Q, V, evidently assume the forms $\mathrm{P}'z^n$, $\mathrm{Q}'z^n$, $\mathrm{V}'z^n$, n being the sum of the dimensions of the variables in each term of the proposed equation. Hence the three equations [4] (427.), become

$$(\mathrm{P}' - \mathrm{V}'t)dz = z\mathrm{V}'dt,$$

$$(\mathrm{Q}' - \mathrm{V}'u)dz = z\mathrm{V}'du,$$

$$\therefore (\mathrm{P}' - t\mathrm{V}')du = (\mathrm{Q}' - u\mathrm{V}')dt.$$

The last being integrated as an equation between the variables t and u, will enable us to eliminate either t or u from one of the preceding equations, which may then be integrated. This being effected, and u and t finally eliminated by $x = tz$ and $y = uz$, we shall obtain the sought integral.

As an example of this process, let the given equation be

$$pxz + qyz = x^2,$$

$$\therefore (1 - t^2)dz = ztdt,$$

$$u(1 - t^2)dz = zt^2du,$$

$$\therefore udt = tdu, \quad \therefore t = \mathrm{M}u, \quad z\sqrt{1 - t^2} = \mathrm{M}',$$

$$\therefore z = \mathrm{M}y, \quad \sqrt{z^2 - x^2} = \mathrm{M}', \therefore z^2 = x^2 + \varphi\left(\frac{x}{y}\right).$$

SECTION XXXII.

Of the integration of partial differential equations of the higher orders.

(436.) A partial differential equation of the nth order, in its most general form, should include, besides the original function and independent variables, all the partial differential coefficients from those of the first to those of the nth order inclusive. It is not consistent with the objects of this work to enter at length into the subject of partial differential equations. We shall, therefore, in the present section, confine ourselves chiefly to differential equations of the second order between three variables, first, however, stating some cases of equations of the higher orders which admit of reduction.

(437.) 1°. Equations between three variables of the form

$$\mathrm{F}\left\{x, y, \frac{d^n z}{dy^n}, \frac{d^{n+1}z}{dx dy^n}, \dots \frac{d^{n+m}z}{dx^m dy^n}\right\} = 0$$

may be reduced to the mth order by putting $v = \dfrac{d^n z}{dy^n}$; for in this case they become

$$\mathrm{F}\left\{x, y, v, \frac{dv}{dx}, \frac{d^2v}{dx^2}, \dots \frac{d^m v}{dx^m}\right\} = 0.$$

Since all the differential coefficients which enter this equation relate to the variation of x, of which v is a function, it may be integrated as an equation between two variables v, x, the variable y being treated as a constant, and introducing m arbitrary functions of y in the integration in place

of the m arbitrary constants. The quantity v being obtained by this process as a function of x and y, the final integral will be obtained by integrating $\frac{d^n z}{dy^n} = v$. In this last integration, x being taken as constant, it will be necessary to introduce n arbitrary functions of x. Thus the complete integral will include m arbitrary functions of y, and n arbitrary functions of x.

(438.) 2°. Equations of the nth order, which include partial differential coefficients with respect to one variable only, may be treated as differential equations between two variables, scil. the function and the variable with respect to which the differentials are taken. In this case, however, in place of introducing arbitrary constants, it will be necessary to introduce arbitrary functions of the remaining variables. Under this case come the two following forms of equations of three variables:

$$F\left(x, y, z, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \dots \frac{d^nz}{dx^n}\right) = 0,$$

$$F\left(x, y, z, \frac{dz}{dy}, \frac{d^2z}{dy^2}, \dots \frac{d^nz}{dy^n}\right) = 0.$$

The equations

$$\frac{d^{n+m}z}{dx^n dy^m} + P\frac{d^nz}{dx^n} = Q,$$

$$\frac{d^{n+m}z}{dx^n dy^m} + P\frac{d^mz}{dy^m} = Q,$$

where P and Q contain no variables, except x and y, also come under this class. For let $\frac{d^nz}{dx^n} = v$, $\therefore$ the former becomes

$$\frac{d^mv}{dy^m} + Pv = Q.$$

And by a similar substitution, the latter assumes the form

$$\frac{d^nv}{dx^n} + Pv = Q.$$

If $m = 1$, the former equation will become

$$\frac{dv}{dy} + \mathrm{P}v = \mathrm{Q},$$

which is of the first order with respect to v and y, and may be integrated according to the rules already given, supplying an arbitrary function of x in place of the constant.

(439.) Before we proceed to consider more general equations, we shall illustrate the preceding cases by some examples.

Ex. 1. Let the proposed equation be

$$\frac{d^2z}{dx^2} = \mathrm{P}\frac{dz}{dx} + \mathrm{Q},$$

where P and Q are functions of x, y, and z. Let $\frac{dz}{dx} = p$, ∴

$$\frac{dp}{dx} = \mathrm{P}p + \mathrm{Q},$$

$$\therefore dp = \mathrm{P}pdx + \mathrm{Q}dx.$$

If $u = \int \mathrm{P}dx$, the integral of this equation is (314.),

$$\frac{dz}{dx} = e^u[\int e^{-u}\mathrm{Q}dx + f(y)],$$

$f(y)$ replacing the arbitrary constant.

Integrating this again, we obtain the sought integral, introducing another arbitrary function of y.

Ex. 2. Let the proposed equation be

$$\frac{d^2z}{dx^2} = \mathrm{Q}, \therefore \frac{dz}{dx} = \int \mathrm{Q}dx + f(y),$$

$$\therefore z = \int dx \int \mathrm{Q}dx + xf(y) + f'(y),$$

$f(y)$ and $f'(y)$ being arbitrary functions of y.

Ex. 3. Let the equation be

$$a\frac{d^2z}{dy^2} = xy,$$

$$\therefore \frac{dz}{dy} = \frac{xy^2}{2a} + f(x),$$

$$\therefore z = \frac{xy^3}{6a} + yf(x) + f'(x).$$

Ex. 4. Let the proposed equation be

$$\frac{d^2z}{dxdy} = \text{M},$$

M being a function of x, y, z. First integrating with respect to y, we obtain

$$\frac{dz}{dx} = \int \text{M}dy + f(x).$$

And integrating with respect to x, we find

$$z = \int dx \int \text{M}dy + \int f(x)dx + f'(y),$$

$f(x)$ and $f'(y)$ being arbitrary functions.

Ex. 5. Let the proposed equation be

$$\frac{d^2z}{dxdy} = ax + by,$$

$$\frac{dz}{dy} = \tfrac{1}{2}ax^2 + byx + f(y),$$

$$\because z = \tfrac{1}{2}ax^2y + \tfrac{1}{2}by^2x + \text{F}(y) + \text{F}'(x),$$

where $\text{F}(y)$, $\text{F}'(x)$, are arbitrary functions.

Ex. 6. Let the proposed equation be

$$\frac{d^2z}{dxdy} = \text{M}\frac{dz}{dx} + \text{N},$$

M and N being functions of x and y. Let $\dfrac{dz}{dx} = p$, $\because$

$$\frac{dp}{dy} = \text{M}p + \text{N}.$$

Integrating this by (314.), we find

$$p = e^u[\text{F}(x) + \int e^{-u}\text{N}dy].$$

Integrating this with respect to x, we find

$$z = \int (e^u dx \int e^{-u}\text{N}dy) + \int e^u \text{F}(x)dx + \text{F}'(y),$$

where $\text{F}(x)$, $\text{F}'(y)$, are arbitrary functions.

Ex. 7. Let the proposed equation be

$$xy\frac{d^2z}{dxdy} = bx\frac{dz}{dx} + ay,$$

$$\because \frac{dz}{dx} = -\frac{ay}{(b-1)x} + y^b\text{F}(x),$$

$$\therefore z = \frac{aybx}{1-b} + y^b F'(x) + f(y).$$

(440.) We shall now proceed to consider the method of integrating partial differential equations of the second order and first degree. The most general equation of this kind is

$$\frac{d^2z}{dx^2}R + \frac{d^2z}{dxdy}S + \frac{d^2z}{dy^2}T = V,$$

where R, S, T, V, are given functions of x, y, z, and the partial differential coefficients of the first order.

Let

$$\frac{dz}{dx} = p, \quad \frac{dz}{dy} = q,$$

$$\frac{d^2z}{dx^2} = r, \quad \frac{d^2z}{dxdy} = s, \quad \frac{d^2z}{dy^2} = t,$$

$$\therefore dp = rdx + sdy, \quad dq = sdx + tdy.$$

By the last two equations, and the general equation

$$rR + sS + tT = V,$$

any two of the three differential coefficients r, s, t, may be eliminated; the third will, however, still remain indeterminate. If r and t be eliminated, the result will be

$$Rdpdy + Tdqdx - Vdxdy = s(Rdy^2 - Sdxdy + Tdx^2).$$

This is simplified by putting

$$dy = mdx,$$

$$\therefore dz = pdx + qmdx,$$

by which substitution, it becomes

$$Rmdp + Tdq - Vmdx = s(Rm^2 - Sm + T) \quad \cdots\cdots [1].$$

Since the quantity s must remain absolutely indeterminate the integral sought must satisfy the conditions

$$Rm^2 - Sm + T = 0 \quad \cdots\cdots [2],$$

$$Rmdp + Tdq - Vmdx = 0 \quad \cdots\cdots [3].$$

If $M = a$, $M' = a'$, be two equations which satisfy these conditions [2], [3], M, M', being functions of x, y, z, p, and q, and a, a', being arbitrary constants, then the equation

$$M = F(M') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [4],$$

in which the form of the function is arbitrary, will be the first integral of the proposed equation. To prove this, it will only be necessary to show that the differential of [4] will always be satisfied by the conditions [2] and [3], independently of the form of the function $F(M')$.

Let the differential of [4] be

$$dM = F'(M')dM',$$

$F'(M')$ being the differential coefficient of $F(M')$ with respect to the variation of M'. Since M, M', are functions of the three variables, and the two first differential coefficients, the total differentials dM, dM', have the forms

$$dM = Adx + Bdy + Cdz + Ddp + Edq,$$
$$dM' = A'dx + B'dy + C'dz + D'dp + E'dq,$$

which, by the substitutions of mdx for dy, and $pdx + qmdx$ for dz, become

$$dM = (A + Bm + Cp + Cqm)dx + Ddp + Edq,$$
$$dM' = (A' + B'm + C'p + C'qm)dx + D'dp + E'dq.$$

Substituting in these the value of dq, derived from [3], they become

$$dM = (A + Bm + Cp + Cqm + \frac{EV}{T}m)dx + \frac{DT - ERm}{T}dp,$$

$$dM' = (A' + B'm + C'p + C'qm + \frac{E'V}{T}m)dx + \frac{D'T - E'Rm}{T}dp.$$

Since by hypothesis the functions M, M', are constant, these differentials must each $= 0$; and since dx and dp are indeterminate, these conditions must be satisfied by their coefficients. Hence we obtain the four equations of condition,

$$T(A + Bm + Cp + Cqm) + EVm = 0,$$
$$T(A' + B'm + C'p + C'qm) + E'Vm = 0,$$
$$DT - ERm = 0,$$
$$D'T - E'Rm = 0.$$

The four quantities A, A′, D, D′, being eliminated by these conditions, and the equation

$$\text{A}dx+\text{B}dy+\text{C}dz+\text{D}dp+\text{E}dq=\text{F}'(\text{M})[\text{A}'dx+\text{B}'dy+\text{C}'dz+ \text{D}'dp+\text{E}'dq],$$

the result, after substituting $pdx + qdy$ for dz, will be

$$(\text{B} + \text{C}q)(dy - mdx) + \frac{\text{E}}{\text{T}} (\text{R}mdp + \text{T}dq - \text{V}mdx) =$$

$$\text{F}'(\text{M}')[(\text{B}' + \text{C}'q)(dy - mdx) + \frac{\text{E}'}{\text{T}} (\text{R}mdp + \text{T}dq - \text{V}mdx)],$$

$$\therefore \text{R}mdp + \text{T}dq - \text{V}mdx = \omega(dy - mdx),$$

where

$$\omega = -\frac{\text{B}+\text{C}q-\text{F}'(\text{M}')(\text{B}'+\text{C}'q)}{\frac{1}{\text{T}}[\text{E}-\text{F}'(\text{M}')\text{E}']}.$$

Substituting $rdx + sdy$ for dp, and $sdx + tdy$ for dq, we shall obtain an equation between the independent differentials dx, dy, which being fulfilled independently of them, will give the equations

$$\text{R}mr + \text{T}s - \text{V}m = - \omega m,$$

$$\text{R}ms + \text{T}t = \omega.$$

Eliminating ω by these equations, we obtain

$$\text{R}mr + \text{R}m^2s + \text{T}s + \text{T}mt - \text{V}m = 0.$$

But by the equation [2]

$$\text{R}m^2 = \text{S}m - \text{T}.$$

Hence we obtain

$$m(\text{R}r + \text{S}s + \text{T}t - \text{V}) = 0,$$

$$\therefore \text{R}r + \text{S}s + \text{T}t - \text{V} = 0,$$

which is the proposed equation.

(441.) Hence we conclude in general, that if the proposed partial differential equation have the form

$$\text{R}r + \text{S}s + \text{T}t - \text{V} = 0;$$

and that either value of m deduced from

$$\text{R}m^2 - \text{S}m + \text{T} = 0$$

being substituted in the equations

$$dy - mdx = 0,$$
$$\text{R}mdp + \text{T}dq - \text{V}mdx = 0;$$
these equations are satisfied by the differentials of the equations
$$\text{M} = a, \quad \text{M}' = b.$$
M and M′ being functions of x, y, z, p, q, the first integral of the proposed equation is
$$\text{M} = \text{F}(\text{M}'),$$
the form of the function being perfectly arbitrary.

Since there are but three equations, viz.
$$dy - mdx = 0,$$
$$\text{R}mdp + \text{T}dq - \text{V}mdx = 0,$$
$$dz = pdx + qdy,$$
between the five variables x, y, z, p, q, the elimination of two will give a differential equation between the remaining three; this, therefore, may not fulfil the criterion of integrability (284.), and the equation, in that case, cannot have a single equation for its integral (418.).

(442.) The following examples will serve as illustrations of these general principles.

Ex. 1. To integrate the equation
$$q^2r - 2pqs + p^2t = 0.$$
In this case,
$$\text{R} = q^2, \quad \text{S} = -2pq, \quad \text{T} = p^2, \quad \text{V} = 0.$$
Hence the equation [2] becomes
$$q^2m^2 + 2pqm + p^2 = 0,$$
$$\therefore qm + p = 0.$$
Eliminating m from the equations $dy - mdx = 0$ and [3], we obtain
$$pdx + qdy = 0, \quad qdp - pdq = 0.$$
Integrating the latter, we find $p = bq$, b being an arbitrary constant. The former gives $dz = 0$, $\therefore z = a$; where a is also an arbitrary constant. Hence the functions M and M′ are in this case $\dfrac{p}{q}$ and z, and therefore the first integral of

the proposed equation is

$$p = q\text{F}(z),$$

where the form of the function is arbitrary. In order to arrive at the primitive equation, it will be necessary to integrate this equation. To effect this, it must be observed, that

$$m = -\frac{p}{q}, \therefore dy = -\text{F}(z)dx,$$

which being integrated, gives

$$y + x\text{F}(z) = \text{F}'(z),$$

which is the primitive equation.

Ex. 2. Let the equation to be integrated be

$$x^2r + 2xys + y^2t = 0.$$

Hence $\text{R} = x^2$, $\text{S} = 2xy$, $\text{T} = y^2$, $\therefore mx - y = 0$, and the equations $dy - mdx = 0$, and [3], become, in this case,

$$ydx - xdy = 0,$$
$$xdp + ydq = 0.$$

The former being integrated, gives

$$y = ax;$$

and eliminating y from the second, we find

$$dp + adq = 0,$$
$$\therefore p + aq = b.$$

Hence the equation $\text{M} = \text{F}(\text{M}')$ becomes

$$px + qy = x\text{F}\left(\frac{y}{x}\right).$$

This being treated by the methods for partial differential equations of the first order, we have

$$dz = \text{F}(a)dx,$$
$$\therefore z = x\text{F}(a) + \text{F}'(a).$$

But since $a = \dfrac{y}{x}$, $\therefore$

$$z = x\text{F}\left(\frac{y}{x}\right) + \text{F}'\left(\frac{y}{x}\right).$$

(443.) If the coefficients of r, s, t, in the equation

$$\text{R}r + \text{S}s + \text{T}t = \text{V}$$

be all constant, and the quantity V be a function of the independent variables alone, the equation [2] becomes a numerical equation, the roots of which are therefore constant. Let these roots be m', m''; which, being substituted in $dy - mdx = 0$ and [3], and the results respectively integrated, give the two systems of equations

$$\left.\begin{array}{l} y - m'x = a \\ \text{R}m'p + \text{T}q - m'\int \text{V}dx = b \end{array}\right\}$$

$$\left.\begin{array}{l} y - m''x = a' \\ \text{R}m''p + \text{T}q - m''\int \text{V}dx = b' \end{array}\right\},$$

in which V may be considered as a function of x alone, since it may be rendered so by substituting $m'x$ or $m''x$ for y.

Hence we have the two first integrals

$$\text{R}m'p + \text{T}q - m'\int \text{V}dx = \text{F}(y - m'x),$$

$$\text{R}m''p + \text{T}q - m''\int \text{V}dx = \text{F}'(y - m''x).$$

By integrating either of these equations, we shall obtain the primitive integral of the proposed equation. If the former be solved for p, we find

$$p = -\frac{\text{T}}{\text{R}m'}q + \frac{\int \text{V}dx}{\text{R}} + \frac{1}{\text{R}}\text{F}(y - m'x).$$

But since by the equation [2] $m'm'' = \dfrac{\text{T}}{\text{R}}$, this becomes

$$p = -m''q + \frac{\int \text{V}dx}{\text{R}} + \frac{1}{\text{R}}\text{F}(y - m'x).$$

Substituting this value of p in the equation

$$dz = pdx + qdy,$$

we obtain

$$\text{R}dz - dx\int \text{V}dx - dx\text{F}(y - m'x) = \text{R}q(dy - m''dx).$$

The equations therefore to be integrated are

$$dy - m''dx = 0,$$

$$\text{R}dz - dx\int \text{V}dx - dx\text{F}(y - m'x) = 0.$$

The former gives $y - m''x = a'$, and the latter becomes

$$\text{R}z - \int dx\int \text{V}dx - \int dx\text{F}(y - m'x) = b.$$

In effecting the integrations indicated in this equation, the following circumstances should be attended to.

1°. In determining $\int \mathrm{v}dx$, y should be replaced in v by $m'x + a$; and after the integration of $\mathrm{v}dx$ has been effected, a should be replaced by $y - m'x$. Then before effecting the integration of $dx \int \mathrm{v}dx$, y should be replaced in $\int \mathrm{v}dx$ by $m''x + a'$, and after the integration has been effected, $y - m''x$ should be substituted for a'.

2°. Before the integration of $dx\mathrm{F}(y - m'x)$, $m''x + a'$ should be substituted for y; and after the integration has been effected, $y - m''x$ should be substituted for a'.

3°. The constant b is an arbitrary function of $y - m''x$.

Hence the complete integral has the form

$$\mathrm{R}z = \int dx \int \mathrm{v}dx + \mathrm{F}(y - m'x) + \mathrm{F}'(y - m''x),$$

where the functions F, F', are arbitrary.

(444.) The following examples will illustrate the preceding formulæ.

Ex. 1. Let the equation be

$$r - s - 2t = \frac{k}{y}.$$

Hence $m^2 + m = 2$, $\because$ $m' = 1$, $m'' = -2$, $\because$ $y = x + a$,

$$y = -2x + a', \because$$

$$\int \mathrm{v}dx = \int \frac{kdx}{x+a} = kl(x+a) = kly,$$

$$\int dx \int \mathrm{v}dx = \int kdxly = \int kdxl(a' - 2x).$$

This becomes, after substituting $2x + y$ for a',

$$\int dx \int \mathrm{v}dx = -kx - kyl\sqrt{y},$$

$$\because z + k(x + yl\sqrt{y}) = \mathrm{F}(y - x) + \mathrm{F}'(y + 2x).$$

Ex. 2. Let the equation be $r - b^2t = 0$, $\because$

$$\frac{d^2z}{dx^2} = b^2\frac{d^2z}{dy^2}.$$

This equation is remarkable, being that of vibrating chords.

In this case $\mathrm{R} = 1$, $\mathrm{S} = \mathrm{V} = 0$, and $\mathrm{T} = -b^2$. Hence

$m' = b$, $m'' = -b$, $\because y = bx + a$, $y = bx + a'$, and $\int dx \int v dx = 0$, $\because$

$$z = \mathrm{F}(y - bx) + \mathrm{F}'(y + bx).$$

(445.) The integration of partial differential equations of the second order is sometimes effected by a process similar to that used in (433.), scil. by the introduction of an indeterminate function θ.

The equation of developable surfaces $rt = s^2$, gives

$$\frac{r}{s} = \frac{s}{t} = \theta,$$

$$\because s = t\theta,$$

$$r = s\theta,$$

$$rdx + sdy = \theta(sdx + tdy), \because dp = \theta dq.$$

This equation is only integrable when θ is a function of q, and in that case the first integral is $p = \mathrm{F}(q)$. The equation

$$dz = pdx + qdy$$

becomes

$$dz = dx\,\mathrm{F}(q) + qdy.$$

Integrating this by the method in (433.), we find, considering q constant,

$$z = x\mathrm{F}(q) + qy + \mathrm{F}'(q),$$

$$0 = xf(q) + y + f'(q),$$

where $f(q)$, $f'(q)$, are the differential coefficients of the functions $\mathrm{F}(q)$, $\mathrm{F}'(q)$.

SECTION XXXIII.

Of the integration of partial differential equations by series.

(446.) By the theorem of Taylor, we are enabled to integrate partial differential equations in series by a method

similar to that explained in Section VI. Let x and y be the independent variables, and z the dependant variable in a differential equation between two variables. Let

$$\mathrm{z}, \ \frac{d\mathrm{z}}{dx}, \ \frac{d^2\mathrm{z}}{dx^2} \cdots \frac{d^n\mathrm{z}}{dx^n} \cdots$$

be what

$$z, \ \frac{dz}{dx}, \ \frac{d^2z}{dx^2} \cdots \frac{d^nz}{dx^n} \cdots$$

become when $x = 0$, and which are therefore functions of y, and the constants which enter z. Hence by Maclaurin's series,

$$z = \mathrm{z} + \frac{d\mathrm{z}}{dx}\frac{x}{1} + \frac{d^2\mathrm{z}}{dx^2}\frac{x^2}{1.2} + \frac{d^3\mathrm{z}}{dx^3}\frac{x^3}{1.2.3} \cdots \quad [1].$$

If it happen that any coefficient when $x = 0$ become infinite, the series may be obtained by substituting $x + a$ for x in the function, and developing by the powers of x.

(447.) If the given partial differential equation be of the first order, let it be solved for either of the partial differential coefficients, so that it will assume the form

$$\frac{dz}{dx} = \mathrm{F}\left(x, y, \frac{dz}{dy}\right).$$

This being differentiated successively with respect to x, and x being supposed $= 0$ in the several coefficients, all the coefficients of the series [1] will be determined as functions of the first term z, which is an arbitrary function of y. Thus the series in this case will include one arbitrary function of y.

If the proposed differential equation be of the second order, let it be

$$\frac{d^2z}{dx^2} = \mathrm{F}\left(\frac{d^2z}{dy^2}, \frac{d^2z}{dxdy}, \frac{dz}{dx}, \frac{dz}{dy}, x, y\right).$$

It is plain that all the quantities which are included in the parenthesis depend on, and can be derived from the

values of z and $\frac{dz}{dx}$, by making $x = 0$ after the operations indicated by the different symbols have been effected, and the subsequent coefficients of the series [1] may be obtained by continued differentiation. The quantities z, $\frac{dz}{dx}$, are in this case arbitrary functions of y, and therefore the complete integral of a partial differential equation of the second order requires the introduction of two arbitrary functions.

By continuing this process, we find, in general, that a partial differential equation of the nth order requires in its complete integral as many arbitrary functions as there are units in n the exponent of its order.

(448.) As an example of integration by series, let the proposed equation of the second order be

$$\frac{d^2z}{dx^2} = c^2\frac{d^2z}{dy^2}.$$

By successive differentiation, we find

$$\frac{d^3z}{dx^3} = c^2\frac{d^3z}{dxdy^2} = c^2\frac{d^2\left(\frac{dz}{dx}\right)}{dy^2},$$

$$\frac{d^4z}{dx^4} = c^2\frac{d^4z}{dx^2dy^2} = c^2\frac{d^2\frac{d^2z}{dx^2}}{dy^2} = c^4\frac{d^4z}{dy^4},$$

$$\frac{d^5z}{dx^5} = c^2\frac{d^5z}{dx^3dy^2} = c^2\frac{d^2\frac{d^3z}{dx^3}}{dy^2} = c^4\frac{d^4\left(\frac{dz}{dx}\right)}{dy^4}.$$

. . . .

. . . .

These several quantities depend only upon the coefficients

$$\frac{dz}{dx},\ \frac{d^2z}{dy^2}.$$

Let $F(y)$ be what z becomes when $x = 0$, and $F''(y)$ be the corresponding value of $\frac{d^2z}{dy^2}$, and let $f(y)$ be what $\frac{dz}{dx}$ becomes when $x = 0$. Hence we find

$$z = F(y) + f(y)\frac{x}{1} + c^2F''(y)\frac{x^2}{1.2} + c^2f''(y)\frac{x^3}{1.2.3}$$
$$+ c^4F''''(y)\frac{x^4}{1.2.3.4} + c^4f''''(y)\frac{x^5}{1.2.3.4.5}\cdots\cdot$$

In this case the functions $F(y)$, $f(y)$, are both arbitrary, but the succeeding coefficients may be derived from them.

The equation proposed may be integrated in finite terms by the rules established in the last section, and its integral will be

$$z = F(y + cx) + f(y - cx).$$

By developing each of these functions, and adding the results, we shall obtain the series already found for z.

(449.) Integrals of partial differential equations may frequently be obtained in series by the method of indeterminate coefficients. This method, used by Lagrange in his *Mecanique Analytique*, consists in assuming a series for z in powers of x, as

$$z = Y + Y'x + Y''x^2 + Y'''x^3 \cdots\cdots [2],$$

which being successively differentiated, gives values for the several differential coefficients. These being substituted in the proposed equation, and the coefficients of the same dimensions of x being equated, the successive coefficients of the above development will be determined. An example will illustrate this. Let the proposed equation be

$$\frac{d^2z}{dx^2} = \frac{dz}{dy}.$$

By differentiating [2] twice for x, and once for y, we obtain

$$\frac{d^2z}{dx^2} = 1.2.\text{Y}'' + 2.3.\text{Y}'''x + 3.4.\text{Y}''''x^2 \dots$$

$$\frac{dz}{dy} = \frac{d\text{Y}}{dy} + \frac{d\text{Y}'}{dy}x + \frac{d\text{Y}''}{dy}x^2 + \dots$$

Equating the coefficients of the corresponding powers of x in these series, we find

$$\text{Y}'' = \frac{d\text{Y}}{dy}\ \frac{1}{1.2}, \quad \text{Y}''' = \frac{d\text{Y}'}{dy}\ \frac{1}{1.2.3},$$

$$\text{Y}'''' = \frac{d\frac{d\text{Y}}{dy}}{dy}\ \frac{1}{1.2.3.4} \dots$$

$$\dots\dots\dots\dots$$

Hence, if $\text{Y} = \text{F}(y)$, and $\text{Y}' = f(y)$, ∴

$$z = \text{F}(y) + f(y)\frac{x}{1} + \text{F}'(y)\frac{x^2}{1.2} + f'(y)\frac{x^3}{1.2.3} \dots$$

If the equation to be integrated be the following between four variables,

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

Let

$$\varphi = \varphi' + \varphi''z + \varphi'''z^2 + \varphi''''z^3 \dots$$

the successive coefficients being functions of x and y.

Differentiating twice for each of the variables, we obtain

$$\frac{d^2\varphi}{dx^2} = \frac{d^2\varphi'}{dx^2} + \frac{d^2\varphi''}{dx^2}z + \frac{d^2\varphi'''}{dx^2}z^2 \dots$$

$$\frac{d^2\varphi}{dy^2} = \frac{d^2\varphi'}{dy^2} + \frac{d^2\varphi''}{dy^2}z + \frac{d^2\varphi'''}{dy^2}z^2 \dots$$

$$\frac{d^2\varphi}{dz^2} = 1.2.\varphi'' + 2.3.\varphi'''z + 3.4.\varphi''''z^2 \dots$$

Substituting these values in the proposed equation, we obtain

$$\frac{d^2\varphi'}{dx^2} + \frac{d^2\varphi'}{dy^2} + 1.2.\varphi''',$$

$$+ \left\{ \frac{d^2\varphi''}{dx^2} + \frac{d^2\varphi''}{dy^2} + 2.3.\varphi'''' \right\} z,$$

$$+\left\{\frac{d^2\varphi'''}{dx^2}+\frac{d^2\varphi'''}{dy^2}+3.4.\varphi^{\mathrm{v}}\right\}z^2.$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

By equating the corresponding coefficients in this and the assumed series, we obtain

$$\varphi''' = -\frac{1}{1.2}\left\{\frac{d^2\varphi'}{dx^2}+\frac{d^2\varphi'}{dy^2}\right\},$$

$$\varphi^{\mathrm{iv}} = -\frac{1}{2.3}\left\{\frac{d^2\varphi''}{dx^2}+\frac{d^2\varphi''}{dy^2}\right\},$$

$$\varphi^{\mathrm{v}} = -\frac{1}{3.4}\left\{\frac{d^2\varphi'''}{dx^2}+\frac{d^2\varphi'''}{dy^2}\right\}$$

$$=\frac{1}{1.2.3.4}\left\{\frac{d^4\varphi'}{dx^4}+2\frac{d^4\varphi'}{dx^2dy^2}+\frac{d^4\varphi'}{dy^4}\right\}.$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Hence we find

$$\varphi=\varphi'+\varphi''\frac{z}{1}-\left(\frac{d^2\varphi'}{dx^2}+\frac{d^2\varphi'}{dy^2}\right)\frac{z^2}{1.2}-\left(\frac{d^2\varphi''}{dx^2}+\frac{d^2\varphi''}{dy^2}\right)\frac{z^3}{1.2.3}\ \cdots\cdots$$

SECTION XXXIV.

Of arbitrary functions.

(450.) The arbitrary functions introduced in the integrals of partial differential equations are analogous to the arbitrary constants introduced in the integrals of exact differentials. The differential equation itself furnishes no data in either case whereby the form of the function, or the value of the constant, may be found. But the differential is generally the result of the analytical statement of some pro-

posed question, and is often more general than the question itself, which may be subject to limitations which the differential equation resulting from it does not express. In these limitations, we sometimes find means for determining the form of the arbitrary function, or the value of the arbitrary constant which enters the solution. Examples, both in geometry and physics, of the determination of the arbitrary constant occur so frequently, that it is not necessary to introduce many here. Let a body be accelerated by an uniform force, commencing to move along the axis of x at the distance x' from the origin, its initial velocity being nothing. The differential equation resulting from this statement will be

$$\mathrm{v}d\mathrm{v} = \mathrm{F}dx,$$

v being the velocity, and F the force.

This being integrated, gives

$$\mathrm{v}^2 = 2\mathrm{F}x + \mathrm{c},$$

c being the arbitrary constant. The differential equation, however, does not include the limitation of the body commencing to move at the distance x'. In order to apply our integral to the solution of the question, it will be therefore necessary to introduce this condition, scil. that $\mathrm{v} = 0$ when $x = x'$, $\because$

$$-2\mathrm{F}x' = \mathrm{c}, \quad \because \mathrm{v}^2 = 2\mathrm{F}(x - x').$$

Thus, the condition of the question, which was not expressed by the differential, is introduced by assigning a proper value to the constant, and therefore serves to determine it.

(451.) The arbitrary functions which are involved in the integrals of partial differential equations are, in particular cases, determined in the same manner.

As an example, let it be required to find the equation of a cylindrical surface generated by the motion of a right line parallel to that whose equations are

$$y = b'z, \quad x = a'z \quad . \quad . \quad . \quad . \quad . \quad [1],$$

and which always passes through an ellipse described upon the plane of xy, and represented by the equation

$$a^2y^2 + b^2x^2 = a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad [2].$$

Let $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$. Since the tangent plane to the proposed cylindrical surface is always parallel to the right line [1], its differential equation is

$$a'\frac{dz}{dx} + b'\frac{dz}{dy} = 1.$$

This equation being then the differential equation of the proposed cylindrical surface, its integral

$$(y - b'z) = \mathrm{F}(x - a'z) \quad . \quad . \quad . \quad . \quad [3],$$

is the equation of the cylindrical surface. In this, $\mathrm{F}(x-a'z)$ is an arbitrary function, and must be determined by the remaining condition of the question. The equation [3] is, in fact, a general equation of all cylindrical surfaces, whose tangent planes are parallel to [1]. In order that the intersection of the cylinder with the plane xy should be the ellipse [2], it will be necessary that [3] be identical with [2] when $z = 0$, $\therefore$

$$\mathrm{F}(x) = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Hence the form of the arbitrary function is determined.

(452.) In like manner, to find the equation of a surface of revolution round the axis of z, of which the generating curve is a parabola, we find the differential equation

$$ydy + xdx = 0,$$

since every section parallel to the plane xy is a circle, z being taken constant. Integrating this, we obtain

$$y^2 + x^2 = \mathrm{F}(z).$$

This is the general equation of surfaces of revolution round the axis of z. The equation of the intersection with the plane zy is

$$y^2 = \mathrm{F}(z);$$

and since this, by supposition, is a parabola, $\therefore$

$$\text{F}(z) = az,$$

$$\therefore y^2 + x^2 = az,$$

is the equation sought.

(453.) In general, then, the arbitrary function of z, which enters the integral, should be determined by some assigned relation between x and y, giving the function z some known form, just as the arbitrary constant is determined by some assigned value of the variable giving the integral some known value.

Thus, let the integral of a partial differential equation have the form

$$\text{MF}(\text{V}) = 1,$$

where M and V express explicit functions of the variables xyz, and $\text{F}(\text{V})$ an arbitrary function. Suppose that it is known from the conditions of the question that the variables satisfy at the same time the two equations

$$\text{F}'(xyz) = 0, \quad f'(xyz) = 0,$$

where the functions are explicitly given. Let $\text{V} = t$, and by this and the two preceding equations, let x, y, and z, be found as functions of t. This being done, substitute those values for xyz in M, and let the result, which will be a function of t, be T. Hence

$$\text{TF}(t) = 1,$$

$$\therefore \text{F}(t) = \frac{1}{\text{T}}.$$

Now, since T is a known function of t, $\therefore$ the form of $\text{F}(t)$ becomes known.

(454.) It frequently occurs, that there are more arbitrary functions than one to be determined. In the following example there are two.

Let the integral be

$$\text{MF}(\text{V}) + \text{NF}'(\text{V}) = 1.$$

In this case, two conditions must be found in the data of the proposed problem, to determine the functions.

Suppose that it is known that the variables satisfy simultaneously

$$F''(xyz) = 0, \quad f''(xyz) = 0;$$

and also,

$$F'''(xyz) = 0, \quad f'''(xyz) = 0.$$

As before, let $V = t$, and by this and the former pair of equations, let xyz be determined as functions of t, and thence M and N determined as functions of t; let these be T and S. Also, let them be determined as functions of t by the latter pair, and let them be T', S'. Hence we have the two equations

$$TF(t) + SF'(t) = 1,$$
$$T'F(t) + S'F'(t) = 1.$$

From which the values of the two functions may be derived as explicit functions of t, and therefore their forms will be known.

This process may easily be generalised and applied to all equations, whatever be the number of arbitrary functions which enter them, provided they be of the form

$$MF(V) + NF'(V) + OF''(V) \cdot \cdot \cdot \cdot = 1.$$

If the integral $MF(V) = 1$ be supposed to be the equation of a curved surface, the supposition that the variables satisfy simultaneously the equations

$$F'(xyz) = 0, \quad f'(xyz) = 0,$$

is equivalent to supposing that the surface passes through a line represented by these equations.

(455.) Very frequently, both in geometrical and physical investigations, the functions are absolutely indeterminate, and remain so. In this case, the results point out general properties, which, without particularising the functions, are common to the whole class included in

the general equation. The function may not even be one which, if represented geometrically, would produce one continued line, but may be represented by any line or combination of lines, however irregular, or may be a curve described *liberâ manu* by no assignable law.

PART III.

THE CALCULUS OF VARIATIONS.

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THE CALCULUS OF VARIATIONS.

SECTION I.

Preliminary Observations and Definitions.

(456.) The method of variations derived its origin from problems in geometry and physics, relative to maxima and minima. An extensive class of such problems, as has been already shown, can be solved with considerable elegance and facility by the application of the principles of the differential calculus. A variety of most interesting questions respecting maxima and minima still, however, remain, and very frequently present themselves in geometrical and physical investigations, to the solution of which the methods established in differential calculus are inadequate.

In general, these methods are only applicable when some maximum or minimum property of a curve or surface of a *given species* is to be determined. But when among all curves or surfaces whatever, which can be drawn under given restrictions, *that species* is sought which possesses some maximum or minimum property, the methods of solution founded on the development of functions, and esta-

blished in Part I, fail. Such questions are solved by the calculus of variations.

These different species of problems respecting maxima and minima will probably be more clearly perceived by examples.

Of the first class are the following:

"Round a given triangle to circumscribe an ellipse, whose area is a minimum."

"In a given triangle to inscribe an ellipse, whose area is a maximum."

In these cases, the species of the sought curve is given, being an ellipse; and such questions can always be solved by the common methods.

The following are examples of the second class of problems before mentioned:

"To find the shortest line which can be drawn connecting two points."

"To find the curve of a given perimeter, which shall enclose the greatest possible area."

"Of all the curves of a given length joining two points, to determine that, which, by its revolution round the right line joining the given points, produces the solid of greatest volume."

The class of problems to which the last two of these examples belong are called *isoperimetrical.* They form one of the principal subjects of investigation which led to the *calculus of variations.* The two BERNOULLI'S, JOHN and JAMES, and TAYLOR, the inventor of the development of functions, were the first who obtained solutions of these problems, and laid thereby the foundation of this science: the methods of investigation gradually improving for a series of years under their hands, were still further advanced by EULER; it was, however, reserved for LAGRANGE to render

it an uniform, systematic, and perfect science, both in principles and notation *.

Those who wish to be informed in the history of the gradual progress and improvement of this interesting department of science without the trouble of tracing it through the volumes of transactions of learned societies, and the various tracts of the Bernoulli's, Taylor, Euler, and Lagrange, will find a very compact account of it in Professor Woodhouse's Tract on Isoperimetrical Problems.

In these latter problems, the *species* of the line is sought, and such investigations are attended with difficulties of a peculiar kind, which we do not meet with in the former class. Problems of this kind occur even more frequently in physics than in geometry. The following are examples of them:

> "To find the line joining two points at different heights, by which a heavy body would descend from the one point to the other in the shortest possible time; or to determine the *brachystochronous curve*."
>
> "To determine the curve of a given length joining two given points, of which the centre of gravity is lowest."
>
> "To determine the solid of least resistance †."

To make the peculiar difficulty attending the solution of such problems apparent, it will be sufficient to reduce one of them to an analytical form. To take one of the simplest, suppose that it is required to draw the shortest possible line between two given points. Let the co-ordinates of the points

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* The name "calculus of variations" was first given to this part of analytical science by Euler.

† This appears to have been one of the earliest problems of this kind. It was proposed by NEWTON in his PRINCIPIA, *lib.* ii. *prop.* 34. *Scholium.*

be $y'x'$, $y''x''$, and let the equation of the line sought be $F(xy) = 0$. The length of the line will be expressed by the integral

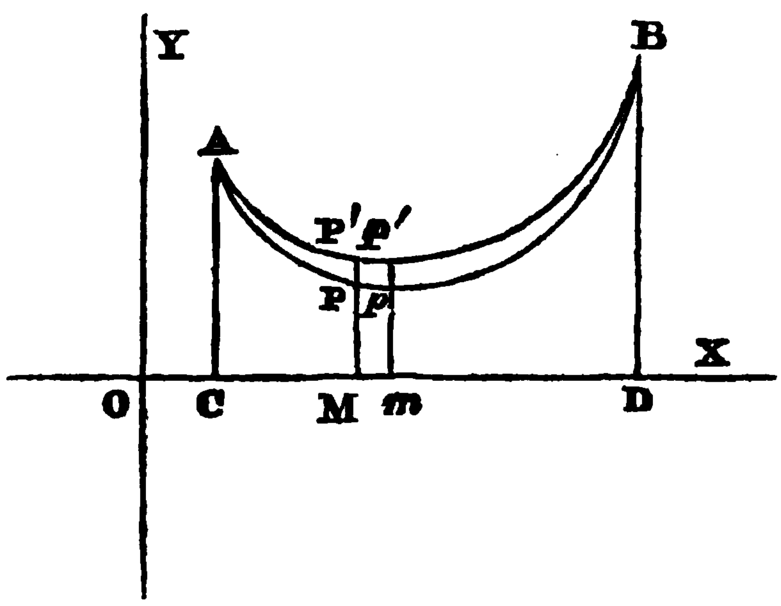
$$U = \int (dy^2 + dx^2)^{\frac{1}{2}},$$

taken between the limits x' and x''. But since the form of the function $F(xy)$ is unknown, dy cannot be eliminated, and therefore the integral is indeterminate, and the integration cannot be effected. We are here required to assign the form of the function $F(xy)$, which will establish such a relation between the variables xy as will render the integral taken between the given limits a minimum.

If such a state of the integral U could be assigned, that every variation in its value consistent with the conditions of the problem would render it greater, that state would be the sought one, and would solve the problem. To determine this state, it must be considered that U varies on two accounts, first, by reason of the variation of the co-ordinates xy of the sought curve; and secondly, by reason of the variation in the form of the function $F(xy)$, which constitutes the relation between these co-ordinates. One of these causes of variation will be removed by assuming the integral between the given limits, for then the co-ordinates of the given points $y'x'$, $y''x''$, will take the place of the variables, and the only cause of variation will be that which depends on the relation between x and y.

(457.) A particular notation has been invented for expressing that variation of x and y which proceeds from a change in the relation between them, which will be most readily apprehended by referring to its geometrical application.

Let A and B be the points whose co-ordinates are $y'x'$, $y''x''$. By a change in the form of the function, let the curve be supposed to change from ApB to Ap'B. Let P be



any point whose co-ordinates are xy. If, while the form of the function remains unchanged, the value of x is increased by Mm, the value of y is changed from PM to pm; and if these changes be assumed of indefinitely small magnitude, they are expressed, as has been explained in the differential calculus, by O$m = x + dx$, $pm = y + dy$. Thus the sign d implies that variation of x and y, which is made on the supposition that the equation $\text{F}(xy) = 0$ remains unchanged, otherwise than by the change in the variables; or, to speak geometrically, differentiation expressed by the character d implies a transition from one point to another of the *same curve*.

Suppose now that the form of the equation $\text{F}(xy) = 0$ undergoes a change. This change producing a change in the relation between x and y, it follows that for each value of x there will be a corresponding value of y different from that value of y which corresponded to the same value of x before the change in the equation. Thus PM being the value of y corresponding to OM before the change, let P′M be the value corresponding to OM $= x$ after the change. Thus we have a variation of y of a kind different from that expressed by dy. This variation of y depending entirely on the change in the equation $\text{F}(xy) = 0$, is usually expressed by δy, and a similar variation of x by δx.

Thus d and δ both signify changes in the variables, the former signifying a change in either produced by a corresponding change in the other, the relation between them being constant; the latter expressing a change in either variable produced by a change in the relation between them,

the other variable being constant. The one is the differential, and the other is called *the variation* of the variable.

In physics, the points of the surface of any body being expressed by xyz referred to three axes of co-ordinates, the variations of xyz by the transition from one point of the surface to another, the position of the surface being unaltered, is expressed by the differentials dy, dx, dz; but a change in any point produced by any motion of the body itself is usually expressed by the *variations* δx, δy, δz. The differentials dx, dy, dz, depend altogether on the figure of the surface, but the variations δx, δy, δz, depend on the *time*, or on some function of it.

(458.) The differential dy being a function derived from the primitive function, is susceptible of variation from the same causes as the primitive function, and the same may be said of $d^2y \cdots$ or of d^ny.

A similar observation applies to the other variables x, z, &c. Hence the meaning of the symbols

$$\begin{array}{lll} \delta dy, & \delta d^2y \cdots & \delta d^ny, \\ \delta dx, & \delta d^2x \cdots & \delta d^nx, \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array}$$

is manifest.

Also the variations δy, δx, &c. being functions of the variables, are susceptible of differentiation.

Hence we perceive the meaning of the expressions

$$\begin{array}{lll} d\delta y, & d^2\delta y \cdots & d^n\delta y, \\ d\delta x, & d^2\delta x \cdots & d^n\delta x. \\ \cdots & \cdots & \cdots \end{array}$$

In the same manner the meaning of the symbols

$$\begin{array}{ll} \delta\int u, & \int\delta u, \\ \delta\int\int u, & \int\int\delta u. \\ \cdots & \cdots \end{array}$$

will be readily apprehended.

From what has been observed, it is plain that the determination of the variation of a function is differentiating it under another point of view, that is to say, ascribing its variation to another cause.

SECTION II.

Of the variation of a function.

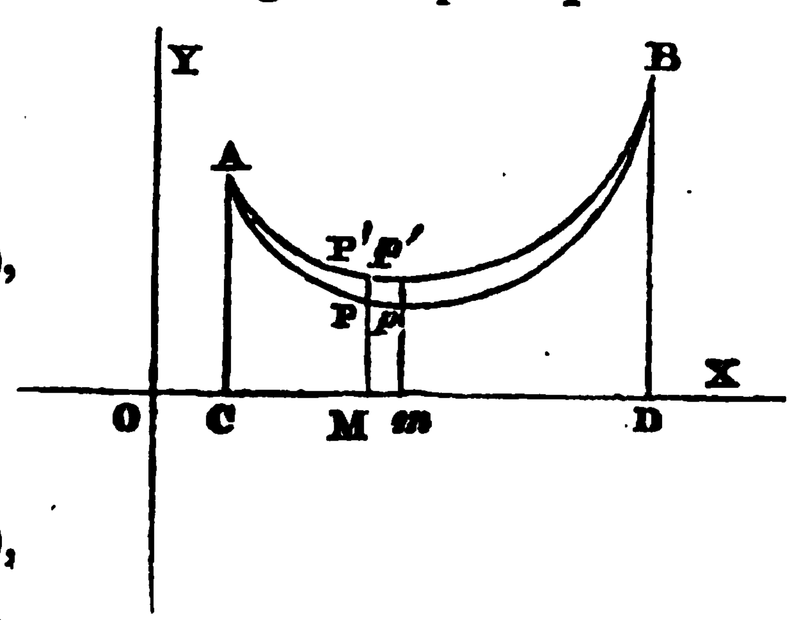
PROP. CIX.

(459.) *In any formula to which* d^n *and* δ *are prefixed, the transposition of these characters does not affect the value of the quantity.*

That is to say,

$$\delta d^n y = d^n \delta y.$$

This might, perhaps, be assumed as true upon the general principle, that when certain given operations are to be performed upon a function, the final result must be the same in whatever order the proposed operations may have been effected. It may, however, be considered satisfactory also to establish it independently of this general principle.



Since

$PM = y$,

$\because pm = y + dy$,

$\because p'm = y + dy + \delta(y + dy)$,

$\because p'm = y + dy + \delta y + \delta dy$.

But also,

$P'M = y + \delta y$,

$\because p'm = y + \delta y + d(y + \delta y)$,

$p'm = y + \delta y + dy + d\delta y$,

$\because \delta dy = d\delta y$.

In general, let $d^{n-1}y = u$, $\therefore$ $d^n y = du$. By what has been just established,

$$\delta du = d\delta u,$$
$$i.\ e.\ \delta d^n y = d^n \delta y.$$

As this is the fundamental principle of the calculus of variations, it may be proper to establish it independently of the consideration of curves.

Let $y = \mathrm{F}(x)$, and when the function by a change in its form becomes $\mathrm{F}'(x)$, we have

$$\delta y = \mathrm{F}'(x) - \mathrm{F}(x).$$

In consequence of the supposed relation between the variables, the difference of these functions must be some function of y. Hence we have

$$\delta y = f(y).$$

Let $y' = y + dy$, $\therefore$

$$\delta y' = f(y'),$$
$$\therefore \delta y + \delta dy = f(y'),$$
$$\therefore \delta dy = f(y') - f(y) = df(y),$$
$$\therefore \delta dy = d\delta y.$$

And hence, in general,

$$\delta d^n y = d^n \delta y.$$

PROP. CX.

(460.) *In any formula to which $\int^n$ and δ are prefixed, the transposition of these characters does not affect the value of the quantity.*

That is,

$$\delta \int^n y = \int^n \delta y,$$

n signifying n successive integrations.

Let $y' = \int^n y$, $\therefore$

$$y = d^n y',$$
$$d^n \delta y' = \delta d^n y'.$$

Taking the nth integral of these, we find

$$\delta y' = \int^n \delta d^n y',$$
$$\text{or } \delta \int^n (y) = \int^n (\delta y).$$

PROP. CXI.

(461.) *To determine the variation of a function of several variables and their successive differentials.*

We shall consider the problem applied to functions of two variables, and the result may then be easily generalised.

Let

$$\text{U} = \text{F}(x, y, x_1, x_2, x_3, \cdots y_1, y_2, y_3, \cdots)$$

where x_1, x_2, x_3, $\cdots$ signify dx, d^2x, d^3x, $\cdots$ and y_1, y_2, y_3, $\cdots$ signify dy, d^2y, d^3y. $\cdots$

Let U become U$'$ when x, y, x_1, y_1, $\cdots$ become $x + \delta x$, $y + \delta y$, $x_1 + \delta x_1$, $y_1 + \delta y_1$, $\cdots$ and by developing, we find

$$\text{U}' - \text{U} = \left\{ \frac{d\text{U}}{dx}\delta x + \frac{d\text{U}}{dy}\delta y + \frac{d\text{U}}{dx_1}\delta x_1 + \frac{d\text{U}}{dy_1}\delta y_1 \cdots \right\}$$
$$+ \left\{ \frac{d^2\text{U}}{dx^2}\delta x^2 + \cdots \right\} + \cdots$$

In variations, as in differentials, the change of the function is supposed indefinitely small, and therefore the terms of this development, in which the powers of the variations which exceed the first enter, may be rejected, and we assume

$$\delta\text{U} = \frac{d\text{U}}{dx}\delta x + \frac{d\text{U}}{dy}\delta y + \frac{d\text{U}}{dx_1}\delta x_1 + \frac{d\text{U}}{dy_1}\delta y_1 + \cdots [1].$$

If there were a greater number of variables, the expression would obviously be similar to this, and hence we derive the following rule:

To find the variation of a function of several variables and their successive differentials, find the several partial differential coefficients of the function with respect to the variables x, y, $\cdots$ *and their successive differentials* x_1, x_2,

$\cdots\, y_1, y_2, \cdots$ *considered as so many independent variables, and multiplying each coefficient by the corresponding variation, the sum of the products is the total variation of the function.*

The method of finding the variation of a function of several independent variables is, therefore, the same as for finding its total differential.

By the result of (458.), combined with the preceding rule, it follows that the variation of a function (U) of several variables (x, y, z, $\cdots$), and their successive differentials, may be expressed thus:

$$\begin{aligned}\delta U = {} & X\delta x + X'\delta dx + X''\delta d^2x \cdots X^{(n)}\delta d^n x,\\ & + Y\delta y + Y'\delta dy + Y''\delta d^2y \cdots Y^{(n)}\delta d^n y,\\ & + Z\delta z + Z'\delta dz + Z''\delta d^2z \cdots Z^{(n)}\delta d^n z,\\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\end{aligned}$$

Where X, X$'$, $\cdots$ Y, Y$'$, $\cdots$ Z, Z$'$, $\cdots$ signify the several partial differential coefficients of U considered as a function of x, dx, $\cdots$ y, dy, $\cdots$ z, dz. $\cdots$

(462.) *Cor.* If U contain only x, y, and the successive differential coefficients of y considered as a function of x, scil.

$$\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \quad \frac{d^3y}{dx^3} = y''', \cdots \frac{d^ny}{dx^n} = y^{(n)},$$

we shall have, as before,

$$\delta U = X\delta x + Y\delta y + Y'\delta y' + Y''\delta y'' \cdots Y^{(n)}\delta y^{(n)},$$

where Y$'$, Y$''$, $\cdots$ are the partial differential coefficients of U considered as a function of y', y''. $\cdots$

(463.) The variations $\delta y'$, $\delta y''$, $\cdots$ may easily be obtained in terms of δy, δx.

$$\delta y' = \delta\frac{dy}{dx} = \frac{\delta dy}{dx} - \frac{dy}{dx}\,\frac{\delta dx}{dx},$$

$$\therefore \delta y' = \frac{\delta dy - y'd\delta x}{dx} = \frac{(d\delta y - y'\delta x)}{dx} + y''\delta x,$$

$$\delta y'' = \frac{d(\delta y' - y''\delta x)}{dx} + y'''\delta x = \frac{d^2(\delta y - y'\delta x)}{dx^2} + y'''\delta x,$$

. . . .

. . . .

$$\delta y^{(n)} = \frac{d^n(\delta y - y'\delta x)}{dx^n} + y^{(n+1)}\delta x.$$

These being substituted in the value of δU, we shall have it expressed in terms of δx and δy.

PROP. CXII.

(464.) *To determine the variation of the integral of a given function of several variables and their differentials.*

Let the function given be U. By (460.),

$$\delta\int \text{U} = \int\delta\text{U}.$$

Let us first suppose U to be a function of two variables x and y. Hence by (461.),

$$\delta\int\text{U} = \int[\text{X}\delta x + \text{X}'\delta dx + \text{X}''\delta d^2x \cdots\cdots],$$
$$+\int[\text{Y}\delta y + \text{Y}'\delta dy + \text{Y}''\delta d^2y \cdots\cdots].$$

This may be modified by the following substitutions suggested by integrating by parts united with the principle

$$\delta d^n y = d^n\delta y:$$

$\int\text{X}\delta x = \int\text{X}\delta x,$

$\int\text{X}'\delta dx = \text{X}'\delta x - \int d\text{X}'\delta x,$

$\int\text{X}''\delta d^2x = \text{X}''d\delta x - \int d\text{X}''d\delta x = \text{X}''d\delta x - d\text{X}''\delta x + \int d^2\text{X}''\delta x,$

$\int\text{X}'''\delta d^3x = \text{X}'''d^2\delta x - d\text{X}'''d\delta x + \int d^2\text{X}'''d\delta x = \text{X}'''d^2\delta x - d\text{X}'''d\delta x$
$+ d^2\text{X}'''\delta x - \int d^3\text{X}'''\delta x.$

. . . .

. . . .

$\int\text{X}^{(n)}\delta d^nx = \text{X}^{(n)}d^{n-1}\delta x - d\text{X}^n d^{n-2}\delta x + d^2\text{X}^n d^{n-3}\delta x \;. \;. \;. \;.$
$d^{n-1}\text{X}^n\delta x \pm \int d^n\text{X}^{(n)}\delta x.$

And, in like manner,

$\int\text{Y}\delta y = \int\text{Y}\delta y,$

$$\int Y'\delta dy = Y'\delta y - \int dY'\delta y,$$
$$\int Y''\delta d^2y = Y''d\delta y - dY''\delta y + \int d^2Y''\delta y,$$
$$\int Y'''\delta d^3y = Y'''d^2\delta y - dY'''d\delta y + d^2Y'''\delta y - \int d^3Y'''\delta y,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\int Y^{(n)}\delta d^ny = Y^{(n)}d^{n-1}\delta y - dY^{(n)}d^{n-2}\delta y + d^2Y^{(n)}d^{n-3}\delta y \ldots\ldots$$
$$d^{n-1}Y^n\delta y \pm \int d^nY^{(n)}\delta y.$$

Making these substitutions in the value of $\delta\int U$, it becomes

$$\begin{aligned}\delta\int U = &\int(X - dX' + d^2X'' - d^3X''' \ldots\ldots)\delta x,\\ &+ (X' - dX'' + d^2X''' - \ldots\ldots\ldots)\delta x,\\ &+ (X'' - dX''' + d^2X^{\mathrm{iv}} - \ldots\ldots\ldots)d\delta x,\\ &+ (X''' - \ldots\ldots\ldots\ldots\ldots\ldots)d^2\delta x,\\ &\cdots\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\cdots\\ &+\int(Y - dY' + d^2Y'' - d^3Y''' + \ldots\ldots)\delta y,\\ &+ (Y' - dY'' + d^2Y''' - \ldots\ldots\ldots\ldots)\delta y,\\ &+ (Y'' - dY''' + d^2Y^{\mathrm{iv}} - \ldots\ldots\ldots\ldots)d\delta y,\\ &+ (Y''' - \ldots\ldots\ldots\ldots\ldots\ldots)d^2\delta y,\\ &\cdots\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\cdots \quad [1].\end{aligned}$$

This value of $\delta\int U$ consists, therefore, of two parts, the one depending on the variation of x, and the other on the variation of y. If there were a greater number of variables involved in the function U, we should have as many more series, and each of them of the same form as the preceding. Thus, if U included the variable z, as well as x and y, we should have, in addition to the above, the following,

$$\begin{aligned}&+\int(Z - dZ' + d^2Z'' - \ldots\ldots)\delta z,\\ &+ (Z' - dZ'' + d^2Z''' - \ldots\ldots)\delta z,\\ &+ (Z'' - dZ''' + \ldots\ldots\ldots)d\delta z,\\ &+ (Z''' - \ldots\ldots\ldots\ldots)d^2\delta z.\\ &\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\end{aligned}$$

It may be observed, that the value of $\delta\!\int\!\text{U}$ will include as many terms affected with the sign $\int$ as there are independent variables.

(465.) *Cor.* The conditions

$$\begin{aligned} \text{X} - d\text{X}' + d^2\text{X}'' \ldots\ldots &= 0, \\ \text{Y} - d\text{Y}' + d^2\text{Y}'' \ldots\ldots &= 0, \\ \text{Z} - d\text{Z}' + d^2\text{Z}'' \ldots\ldots &= 0, \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & \quad [2], \end{aligned}$$

leave the variation of $\int\text{U}$ free from any integral sign.

These are the conditions which determine the function U to be an exact differential, or to be integrable. The general process for establishing this criterion by the principles of the integral calculus is not sufficiently elementary to render its introduction in the second part of this work proper. We shall here, however, establish the conditions by the method of variations.

Let U be supposed to be the exact differential of a function U_1, and let $\text{U} = d\text{U}_1$, ∵

$$\delta\text{U} = \delta d\text{U}_1 = d\delta\text{U}_1.$$

It appears from this, that if U be a complete differential, δU will also be one. Hence, when all those terms which are susceptible of integration shall be brought from under the sign $\int$, those which remain must, collectively, $= 0$, independently of any relations between the variables or their variations.

PROP. CXIII.

(466.) *To determine the variation of the integral of a given differential* $\text{V}dx$, *when* V *is a function of several variables, and their differential coefficients considered as functions of one common variable* x.

As before, we shall first consider the problem applied to

two variables. Let v then be a function of $y, x, y', y'' \ldots$ where

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}, \ldots$$

In the present case, $\text{U} = \text{v}dx$, $\therefore$ (460.),

$$\delta\int \text{v}dx = \int\delta(\text{v}dx) = \int \text{v}d\delta x + \int dx\delta \text{v}.$$

But

$$\int \text{v}d\delta x = \text{v}\delta x - \int d\text{v}\delta x,$$

$$\therefore \delta\int \text{v}dx = \text{v}\delta x + \int(dx\delta\text{v} - d\text{v}\delta x).$$

The values of δv and $d\text{v}$ are, (462.),

$$\delta\text{v} = \text{X}\delta x + \text{Y}\delta y + \text{Y}'\delta y' + \text{Y}''\delta y'' \ldots$$

$$d\text{v} = \text{X}dx + \text{Y}dy + \text{Y}'dy' + \text{Y}''dy'' \ldots$$

where $\text{Y}', \text{Y}'', \ldots$ signify $\frac{d\text{v}}{dy'}, \frac{d\text{v}}{dy''} \ldots$

By which substitutions, we obtain

$$dx\delta\text{v} - d\text{v}\delta x = \text{Y}(dx\delta y - dy\delta x) + \text{Y}'(dx\delta y' - dy'\delta x) + \text{Y}''(dx\delta y'' - dy''\delta x) + \ldots$$

By the values of $\delta y', \delta y'', \ldots$ found in (463.), we obtain

$$dx\delta y - dy\delta x = dx(\delta y - y'\delta x) = u dx,$$

$$dx\delta y' - dy'\delta x = d(\delta y - y'\delta x) = du,$$

$$dx\delta y'' - dy''\delta x = \frac{d^2(\delta y - y'\delta x)}{dx} = d\frac{du}{dx},$$

$$\cdots\cdots\cdots\cdots\cdots$$

where $u = \delta y - y'\delta x$.

Hence we obtain

$$\int(dx\delta\text{v} - d\text{v}\delta x) = \int \text{Y}u dx + \int \text{Y}'du + \int \text{Y}''d\frac{du}{dx} \ldots$$

Integrating by parts each term which contains differentials of u, we find

$$\int \text{Y}'du = \text{Y}'u - \int\frac{d\text{Y}'}{dx}u dx,$$

$$\int \text{Y}''d\frac{du}{dx} = \text{Y}''\frac{du}{dx} - \int\frac{d\text{Y}''}{dx}du = \text{Y}''\frac{du}{dx} - \frac{d\text{Y}''}{dx}u + \int\frac{1}{dx}d\frac{d\text{Y}''}{dx}u dx,$$

$$\cdots\cdots\cdots\cdots\cdots$$

Hence we find

$$\delta\int \mathrm{v}dx = \mathrm{v}\delta x + \left\{ \mathrm{Y}' - \frac{d\mathrm{Y}''}{dx} + \ldots \right\} u,$$
$$+ \left\{ \mathrm{Y}'' - \ldots\ldots\ldots \right\} \frac{du}{dx},$$
$$+ \ldots\ldots\ldots\ldots$$
$$+ \ldots\ldots\ldots\ldots$$
$$+ \int \left\{ \mathrm{Y} - \frac{d\mathrm{Y}'}{dx} + \frac{1}{dx} d\frac{d\mathrm{Y}''}{dx} - \ldots \right\} u dx.$$

This is the variation sought.

(467.) *Cor.* Since $u = \delta y - y'\delta x$, it appears that the coefficients of δy and δx, under the sign of integration, have a common factor; and it therefore follows that the same condition will make them both vanish, and leave the variation independent of any integral. This condition is evidently

$$\mathrm{Y} - \frac{d\mathrm{Y}'}{dx} + \frac{1}{dx} d\frac{d\mathrm{Y}''}{dx} - \ldots = 0.$$

From what has been already observed, it is plain that this is the condition which determines $\mathrm{v}dx$ to be integrable.

SECTION III.

On the maxima and minima of indeterminate integrals.

(468.) We shall now proceed to the investigation of the class of maxima and minima problems already mentioned, and to which the methods explained in the differential calculus do not reach. These problems, when reduced to an analytical statement, generally come under the following form:

"Given a differential expression U between any variables
"and their differentials to assign that relation between

"the variables for which the integral of the proposed "expression taken between any assigned limits will "have a maximum or minimum value."

To apply the method explained in the differential calculus, it would be necessary to know the form of the integral; whereas, in the present case, the form is the thing sought, and must be deduced from the very circumstance of the integral being a maximum or minimum.

If the problem be geometrical, the integral, whose maximum is sought, usually expresses some quantity depending on a curve or surface. Thus the integrals

$$\int\sqrt{dy^2 + dx^2}, \quad \int y dx,$$

although really indeterminate, since no relation is given between x and y, yet express quantities depending on the sought curve, the former signifying its length, the latter its area.

In like manner, if the question be physical, the indeterminate integral may express the time, velocity, force, &c. the maximum or minimum of which is sought.

The principles of variations already established, however, will enable us to extend the method for finding the maxima and minima of determinate functions to indeterminate integrals.

(469.) Let u be the indeterminate function of which the maximum or minimum is sought, and let u' be what this becomes when $x, y, dx, dy, \ldots$ are changed into $x + \delta x$, $y + \delta y$, $dx + \delta dx$, $dy + \delta dy \ldots$ In order that u may be a maximum or minimum, it is necessary that the sign of $\text{u}' - \text{u}$ may be independent of the signs of the increments $\delta x, \delta y \ldots$ Hence the term which involves the first powers of these must $= 0$, $\because \delta\text{u} = 0$. Thus, that the indeterminate function may be a maximum or minimum, it is necessary that its variation should vanish. This condition is *necessary*, but not *sufficient*. Besides this, it is required

that the terms involving the increments in two dimensions should collectively, as to sign, be independent of δx, δy, ... and hence all the circumstances incident on common maxima and minima of functions of several variables are also to be attended to here.

PROP. CXIV.

(470.) *To determine the relation between the variables which will render an indeterminate integral taken between assigned limits a maximum or minimum.*

If the proposed indeterminate integral be $\int \mathrm{U}$, it is necessary that $\delta \int \mathrm{U} = 0$. Assuming the value [1] of this, determined in (464.), it is necessary that this should $= 0$. This value consists of very distinct parts, some affected by the sign of integration $\int$, others free from it. Since the variations δx, δy, ... are supposed to be independent, the terms affected by the sign $\int$ are integrable, and, therefore, of the whole value of $\delta \int \mathrm{U} = 0$, those parts which are affected by the sign $\int$ must separately $= 0$; for, otherwise, they would be equal to the remaining part, and would be therefore integrable. Hence the condition $\delta \int \mathrm{U} = 0$ requires that the system of equations [1] and [2] should be both satisfied.

The number of equations in the system [2] is, in general, equal to that of the independent variations. In case, however, of but two variables, U assuming the form $\mathrm{V}dx$, these equations may be reduced to one (467.).

The conditions [2] reduce [1] to the form

$$\begin{aligned} \int \delta \mathrm{U} = {} & \mathrm{X}'\delta x + \mathrm{X}''d\delta x + \mathrm{X}'''d^2\delta x, \cdots \\ & + \mathrm{Y}'\delta y + \mathrm{Y}''d\delta y + \mathrm{Y}'''d^2\delta y, \cdots \\ & \cdots\cdots\cdots\cdots \end{aligned}$$

where X', X'', ... Y', Y'', ... signify the quantities in-

cluded in the parentheses in [1]. Let the values of the variables corresponding to the limits of the integral be $x'y'z' \ldots\ldots x''y''z'' \ldots\ldots$ and when these are substituted for the variables, let the values of the integral $\int \delta\text{U}$ become L' and L''. Hence

$$\int \delta\text{U} = \text{L}'' - \text{L}';$$

and, therefore,

$$\text{L}'' - \text{L}' = 0$$

is a condition of the proposed maximum or minimum.

This equation will then contain no variables, except those which correspond to the limits, which, however, may or may not be variable according to the conditions which regulate the proposed limits. The system of equations [2] express the sought relation between the variables. If the problem be geometrical, these will be the equation of the curve or surface sought, observing, however, that it is to be modified by the conditions of $\text{L}'' - \text{L}' = 0$, and the relation between the proposed limits. The process of solution will be more readily perceived by considering successively the different conditions which may affect the limits of the integral, and illustrating these conditions by their geometrical application.

(471.) 1°. If the limits of the proposed integral are absolutely given and fixed. In this case, $x'y'z' \ldots\ldots x''y''z'' \ldots\ldots$ being supposed to be the particular values of the variables corresponding to the limits, are fixed, and subject to no variation. Hence, in L'' and L' we must put $\delta x' = 0$, $d\delta x' = 0$, $\ldots\ldots \delta y' = 0$, $d\delta y' = 0 \ldots\ldots$ and since these quantities enter every term of L'' and L' as factors, the condition $\text{L}'' - \text{L}' = 0$ will be fulfilled independently of the coefficients. In this case the relation between the variables is found by integrating the system of equations [2], and ascribing such values to the arbitrary constants, that the integral will satisfy the conditions of the proposed limits.

Thus, in geometry, if the curve sought, and which must

have the proposed maximum or minimum property, is also required to pass through two given points, the co-ordinates of these points determine the limits of the integral.

The equations [2] being integrated, give the general equation of the curve sought; but it will be necessary to assign such values to the arbitrary constants introduced in the integration, that the curve shall pass through the two given points.

(472.) 2°. If the limits be absolutely arbitrary and independent, it is necessary that the equation $L'' - L' = 0$ shall be fulfilled by its coefficients; that is, that the coefficient of each variation in it shall separately $= 0$.

(473.) 3°. If the values of the variables corresponding to the limits be subject to any conditions expressed by equations, these equations will give, by differentiation, relations between the variations of the variables corresponding to the limits. As many variations may be eliminated from $L'' - L' = 0$ as there are independent equations of condition. The remaining variations being absolutely arbitrary and independent, the resulting equation must be fulfilled by its coefficients; that is, the coefficient of each remaining variation must separately $= 0$.

(474.) The same may be effected upon another principle. Let $u = 0$ and $v = 0$ be the equations by which the particular values of the variables corresponding to the limits are restricted.

Hence the conditions $\delta u' = 0$, $\delta u'' = 0$, must subsist at the same time with $L'' - L' = 0$. These three equations may be expressed by one, thus,

$$L'' - L + A'\delta u' + A''\delta u'' = 0,$$

the coefficients A', A'', being supposed to be arbitrary constants entirely independent of $L'' - L'$, $\delta u'$, or $\delta u''$. This supposition evidently renders the one equation equivalent to the three former, for, otherwise, it would express a relation

between the quantities A', A'', and the quantities $L'' - L'$, $\delta u'$, and $\delta u''$, which is contrary to hypothesis.

This principle is of very extensive use in the application of the calculus of variations to, geometry and physics. In place of eliminating the dependant variations, we treat them as independent quantities in the above equation, and equate each of their coefficients with 0, and from the equations thus resulting, the arbitrary quantities A', A'', ... being eliminated, the result, which will be obtained, will be equivalent to that which would have been found by eliminating the variations by the equations of condition. The method which we have now explained is, however, in most cases preferable.

(475.) Thus, in geometry, if the curve sought be not as before restricted to terminate in two fixed points, but only to terminate in two given curves or surfaces: in this case, the co-ordinates of the limits are only restricted to satisfy the equations of the limiting curves or surfaces. In this case, the variations of the co-ordinates at the limits must be related to each other in the same manner as the differentials of the co-ordinates of the given curves or surfaces. These conditions being introduced into $L'' - L' = 0$ by elimination, as already described, the coefficients of those independent variations which remain must be put separately $= 0$.

Again, the limits may be still further restricted. Let the sought curve be not only required to be terminated in given curves or surfaces, but also to touch them. In this case it will not be enough that the co-ordinates of the limits satisfy the equations of the limiting curves or surfaces, but the differentials of the co-ordinates must also satisfy them. Hence the variations of the differentials of these co-ordinates must be equivalent to the second differentials of the co-ordinates of the limiting surfaces. By these conditions, the number of variations which may be eliminated are increased, and

the independent equations involved in $L'' - L' = 0$ are therefore diminished.

In these cases, as in the first, the constants introduced in the integration of [2] must be so assumed as to satisfy the equations resulting from $L'' - L' = 0$.

(476.) From the form of the differential equations [2], it is evident that their *order* may be marked by any number not exceeding twice that which characterises the formula U, and therefore the integral may involve any number of arbitrary constants not exceeding this.

The number of terms in the equation $L'' - L' = 0$ increases with the order of the formula U, and, therefore, with the number of arbitrary constants in U. In general, then, the higher the order of the formula U, the greater number of conditions we are at liberty to impose upon the limits; these conditions being always satisfied by the values ascribed to the arbitrary constants in the integrals of [2].

(477.) When the co-ordinates of the limits are variable, as in the cases last mentioned, and enter the formula U, which sometimes happens in taking the variation of U, these co-ordinates are to be considered as independent variables, and their variations must enter the total variation of U. But, in integrating δU with respect to the variables $x, y, z,$ .... the co-ordinates of the limits, and their variations, are to be regarded as constants, and brought outside the sign of integration, so that any term of the form $\int A\delta x'$ may be replaced by $\delta x'\int A$. This is evident, since the integration may be conceived to respect the variation of xyz .... through the sought curve, and not from one of its positions to another. An instance of the necessity of attending to this circumstance occurs in investigating the *brachystochronous curve*.

(478.) It sometimes happens that the variations δx, δy, δz, ... are restricted by equations of condition altogether independent of the limits of the integral. Thus, for ex-

ample, when the curve sought is required to be drawn upon a given curved surface, as in the case of finding the shortest distance between two points upon a given surface.

Since in this case a relation subsists between the variations, it is not necessary, in order that the integral sign should disappear from the value of $\delta\!\int\!\mathrm{U}$, that the several terms which it affects should severally $= 0$. The number of these terms may be diminished by eliminating as many of the variations as there are independent equations of condition given, and then putting the coefficients of the remaining variations $= 0$. The number of equations in the system [2] will, in this case, not be equal to the number of variables, but to the number of independent variations.

SECTION IV.

Examples on the calculus of variations.

PROP. CXV.

(479.) *To find the shortest line between two points.*

In this case

$$\int \mathrm{U} = \int \sqrt{dx^2 + dy^2 + dz^2} = \int ds,$$

$$\therefore \delta \mathrm{U} = \frac{dx}{ds}\delta dx + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta dz.$$

Comparing this with the formula for $\delta \mathrm{U}$ in (461.), we find

$$\mathrm{X} = 0, \quad \mathrm{Y} = 0, \quad \mathrm{Z} = 0,$$

$$\mathrm{X}' = \frac{dx}{ds}, \quad \mathrm{Y}' = \frac{dy}{ds}, \quad \mathrm{Z}' = \frac{dz}{ds},$$

and all the other coefficients $= 0$.

The system of equations [2] become

$$d\left(\frac{dx}{ds}\right) = 0, \quad d\left(\frac{dy}{ds}\right) = 0, \quad d\left(\frac{dz}{ds}\right) = 0,$$

$$\because dx = a\,ds, \quad dy = b\,ds, \quad dz = c\,ds,$$

a, b, c, being arbitrary, subject, however, to the condition

$$a^2 + b^2 + c^2 = 1.$$

Eliminating ds, and integrating the resulting equations between dx, dy, dz, we find two equations of the forms

$$z = \text{A}x + \text{B},$$
$$z = \text{A}'y + \text{B}',$$

which are equations of a right line. Since the limits of the integral are absolutely fixed, the equation $\text{L}'' - \text{L}' = 0$ is necessarily fulfilled; so that all which remains to complete the solution is, to subject the right line to pass through the two given points, by which condition, the arbitrary constants A, B, A$'$, B$'$, are determined.

Let $x'y'z'$, $x''y''z''$, be the co-ordinates of the limits.

The equations become

$$z - z' = \frac{z''-z'}{x''-x'}(x - x'),$$

$$z - z' = \frac{z''-z'}{y''-y'}(y - y'),$$

which are the equations of the line sought.

PROP. CXVI.

(480.) *To find the shortest line between two surfaces, of which the equations are given.*

The solution of this problem is precisely the same as the last, except in the elimination of the arbitrary constants. Let the equations of the two limiting surfaces be $z' = \text{F}(x'y')$, $z'' = f(x''y'')$, $\because$

$$\delta z' = p'\delta x' + q'\delta y',$$
$$\delta z'' = p''\delta x'' + q''\delta z''.$$

Since the value of $\delta \int U$ is, in this case,

$$\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z = 0,$$

$$\text{or } dx\delta x + dy\delta y + dz\delta z = 0.$$

Substituting successively for the variables $x'y'z'$, $x''y''z''$, and eliminating $\delta z'$, $\delta z''$, we find

$$\text{L}'' - \text{L}' = (dx' + p'dz')\delta x' + (dy' + q'dz')\delta y'$$
$$- (dx'' + p''dz'')\delta x'' - (dy'' + q''dz'')\delta y'' = 0.$$

Since the variations here are independent, their coefficients must severally vanish, ∴

$$dx' + p'dz' = 0, \quad dy' + q'dz' = 0,$$
$$dx'' + p''dz'' = 0, \quad dy'' + q''dz'' = 0.$$

But by differentiating the equations of the line, we find

$$\frac{dz}{dx} = \text{A}, \quad \frac{dz}{dy} = \text{A}'.$$

Hence

$$\text{A} = -\frac{1}{p'}, \quad = -\frac{1}{p''},$$
$$\text{A}' = -\frac{1}{q'}, \quad = -\frac{1}{q''}.$$

From which it is evident, that the line must be a normal to both the given surfaces.

(481.) *Cor.* 1. If the line were drawn from a fixed point to one of the surfaces, it would in like manner follow, that it must be normal to that surface.

(482.) *Cor.* 2. A process exactly similar will show that the shortest line which can be drawn between two curves in the same plane is their common normal.

PROP. CXVII.

(483.) *To find the equation of the shortest line joining two points upon a given curved surface.*

In this case the variations δx, δy, δz, are connected by an equation found by differentiating under the character δ, the equation $u = 0$ of the given surface; ∵ (461.)

$$\frac{du}{dx}\delta x + \frac{du}{dy}\delta y + \frac{du}{dz}\delta z = 0;$$

and since the values of the coefficients x, &c. are the same as in the former propositions, we have by the equation [1] (464.)

$$dx'\delta x + dY'\delta y + dz'\delta z = 0;$$

$$\text{or } d\left(\frac{dx}{ds}\right)\delta x + d\left(\frac{dy}{ds}\right)\delta y + d\left(\frac{dz}{ds}\right)\delta z = 0.$$

Eliminating δz by this and the first, we find

$$\left\{\frac{du}{dx}d\left(\frac{dz}{ds}\right) - \frac{du}{dz}\left(\frac{dx}{ds}\right)\right\}\delta x +$$
$$\left\{\frac{du}{dy}d\left(\frac{dz}{ds}\right) - \frac{du}{dz}d\left(\frac{dy}{ds}\right)\right\}\delta y = 0.$$

Since δy and δx are here independent, we have

$$\frac{du}{dx}d\left(\frac{dz}{ds}\right) - \frac{du}{dz}d\left(\frac{dx}{ds}\right) = 0,$$
$$\frac{du}{dy}d\left(\frac{dz}{ds}\right) - \frac{du}{dz}d\left(\frac{dy}{ds}\right) = 0,$$

which are the equations of the curve sought. It will be necessary, in integrating these, to eliminate the arbitrary constants by the conditions of the curve passing through the given points.

If, for example, the surface be a sphere, of which the origin is the centre, we have

$$\frac{du}{dx} = 2x, \quad \frac{du}{dy} = 2y, \quad \frac{du}{dz} = 2z.$$

If ds be considered constant, we find by this

$$xd^2z - zd^2x = 0, \quad yd^2z - zd^2y = 0,$$
$$\therefore yd^2x - xd^2y = 0.$$

Integrating these, we have

$$xdz - zdx = ads, \quad ydz - zdy = bds,$$
$$ydx - xdy = cds.$$

Multiplying by y, x, and z, respectively, and adding, we find, after expunging the common factor ds,

$$ay + bx + cz = 0,$$

which is the equation of a plane passing through the centre of the sphere. In this plane, therefore, the sought curve must be, and since it is also on the surface of the sphere, it is evident that it must be the arc of a great circle.

PROP. CXVIII.

(484.) *To find that curve of a given length drawn between two given points which will include with its extreme ordinates and the intercept of the axis of* X *between them the greatest possible area.*

In this case $\text{U} = \int ydx$ and $\int ds$ is constant, $\therefore$ the conditions of the question are expressed by the equations

$$\delta\int \text{U} = \delta\int ydx = 0,$$
$$\delta\int ds = 0;$$

or if A be an arbitrary constant, these two equations are included in (474.),

$$\delta\int ydx + \text{A}\delta\int ds = 0,$$
$$\delta\int ydx = \int(\delta ydx + y\delta dx),$$
$$\delta\int ds = \int\delta(dy^2 + dx^2)^{\frac{1}{2}} = \int\frac{dy\delta dy + dx\delta dx}{ds},$$
$$\therefore \int(y\delta dx + dx\delta y + \frac{\text{A}dx\delta dx + \text{A}dy\delta dy}{ds}) = 0.$$

Hence by comparing this with the result of (464.), we find

$$\mathrm{X} = 0, \quad \mathrm{X}' = y + \mathrm{A}\frac{dx}{ds},$$

$$\mathrm{Y} = dx, \quad \mathrm{Y}' = \mathrm{A}\frac{dy}{ds}.$$

Hence the equations [2] (465.) become

$$d\left(y + \mathrm{A}\frac{dx}{ds}\right) = 0,$$

$$dx - \mathrm{A}d\frac{dy}{ds} = 0,$$

which being integrated, give

$$y + \mathrm{A}\frac{dx}{ds} = c, \quad x - \mathrm{A}\frac{dy}{ds} = c'.$$

These equations being integrated, would be identical; they will not, therefore, serve to eliminate A. Substituting $\sqrt{dy^2 + dx^2}$ for ds, in the first we find

$$\frac{dy}{dx} = \frac{\sqrt{\mathrm{A}^2 - (y-c)^2}}{y-c},$$

$$\therefore (x - c')^2 + (y - c)^2 = \mathrm{A}^2.$$

The curve is therefore a circle, and the result will be a maximum or minimum, according as the concavity is turned towards or from the axis of x. The co-ordinates c, c', of the centre, and the arbitrary A^2, must be determined by the two points through which the circle is required to pass, and the length of the arc between them.

From this it is obvious, that of all isoperimetrical figures, the circle includes the greatest area, and also, that of all figures including a given area, the circle has the least perimeter.

PROP. CXIX.

(485.) *Of all solids of revolution having equal surfaces included between given limits, to determine that which has the greatest volume.*

The conditions of this question give

$$\pi\int y^2 dx = maximum,$$
$$2\pi\int y ds = constant.$$

Hence, by the same principles as those used in the last proposition, we have

$$\int\left(2y dx\delta y + y^2\delta dx + 2\text{A}y\frac{dx\delta dx + dy\delta dy}{ds} + 2\text{A}ds\delta y\right) = 0,$$

which gives

$$\text{X} = 0, \qquad \text{X}' = 2\text{A}y\frac{dx}{ds} + y^2,$$

$$\text{Y} = 2y dx + 2\text{A}ds, \quad \text{Y}' = 2\text{A}y\frac{dy}{ds}.$$

Hence the equations [2] become

$$d(2\text{A}y\frac{dx}{ds} + y^2) = 0,$$

$$y dx + \text{A}ds - \text{A}d\left(y\frac{dy}{ds}\right) = 0.$$

The former gives

$$2\text{A}y\frac{dx}{ds} + y^2 = c,$$

which gives

$$dx = \frac{(c-y^2)dy}{\sqrt{4\text{A}^2y^2-(c-y^2)^2}}.$$

The integral of which, assumed within any proposed limits, will give the generatrix of the solid sought.

If $c = 0$, the equation to be integrated becomes

$$dx = \frac{-ydy}{\sqrt{4A^2-y^2}},$$

$$\therefore x = \sqrt{4A^2-y^2}+c',$$

which is the equation of a circle, whose centre is on the axis of x. The values of A and c' are to be determined by the limits of the proposed solid. These limits may be supposed to be given by the distances of the planes which bound the solid perpendicular to its axis from the origin of co-ordinates.

PROP. CXX.

(486.) *Of all plane curves of a given length drawn joining two given points, to determine that which produces by its revolution the solid of greatest volume.*

In this case

$$\int U = \pi \int y^2 dx,$$

$$\therefore \delta \int U = \pi \int (y^2 \delta dx + 2y dx \delta y).$$

But since the curve has a given length,

$$\int (dy^2 + dx^2)^{\frac{1}{2}} = constant,$$

$$\therefore \int \frac{dy\delta dy + dx\delta dx}{ds} = 0$$

Multiplying this by the arbitrary coefficient A and adding it to the former, we obtain

$$\int [2\pi y dx \delta y + (\pi y^2 + A\frac{dx}{ds})\delta dx + A\frac{dy}{ds}\delta dy] = 0.$$

Hence

$$X = 0, \qquad X' = \pi y^2 + A\frac{dx}{ds},$$

$$Y = 2\pi y dx, \quad Y' = A\frac{dy}{ds}.$$

By integrating the first of [2], after these substitutions, we obtain

$$\pi y^2 + A\frac{dx}{ds} = c,$$

which being equivalent to the second, A cannot be eliminated. This equation, by eliminating ds, becomes

$$dx = \frac{(c-\pi y^2)dy}{\sqrt{(A^2-c-\pi y^2)}}.$$

This is the equation of the *elastic curve*. There will be three arbitrary constants in the primitive equation, viz. A, c, and the constant introduced in the final integration. These three constants will be determined by the two given points and the given length of the curve.

PROP. CXXI.

(487.) *Of all plane curves of a given length drawn between two given points, to determine that which by its revolution produces the solid of greatest surface.*

In this case we have

$$\int\sqrt{dy^2 + dx^2} = constant,$$

$$\int 2\pi y\sqrt{dy^2 + dx^2} = max.$$

Taking the variations of these, we have

$$\int\frac{dy\delta dy + dx\delta dx}{ds} = 0,$$

$$\int 2\pi\left\{ds\delta y + y\frac{dy\delta dy + dx\delta dx}{ds}\right\} = 0.$$

Multiplying the former by the indeterminate constant A, and adding, we find

$$\int\left\{2\pi ds\delta y + (2\pi y + A)\left(\frac{dy}{ds}\delta dy + \frac{dx}{ds}\delta dx\right)\right\} = 0,$$

$$\therefore X = 0, \qquad X' = (2\pi y + A)\frac{dx}{ds},$$

$$Y = 2\pi ds, \quad Y' = (2\pi y + A)\frac{dy}{ds},$$

$$\therefore \frac{2\pi y dx + A dx}{ds} = c.$$

This is the differential equation of a catenary, which gives the maximum or minimum, according as its concavity is turned from or towards the axis.

PROP. CXXII.

(488.) *Two points are given at different perpendicular distances from the horizon, to find the line of swiftest descent from the one to the other, or the brachystochronous curve joining them.*

The origin being assumed at the higher point, and the axis of y vertical, the velocity of the body may be represented by $\sqrt{y}$, and, therefore, the time will be $\int \frac{ds}{\sqrt{y}}$. Hence we must find, when this integral taken between the proposed limits is a minimum. Since

$$ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

In this case,

$$\delta \int \frac{ds}{\sqrt{y}} = \int \left\{ \frac{dx\delta dx + dy\delta dy + dz\delta dz}{\sqrt{y}ds} - \frac{ds}{2y^{\frac{3}{2}}}\delta y \right\}.$$

Hence

$$\text{X} = 0, \qquad \text{X}' = \frac{dx}{ds\sqrt{y}},$$

$$\text{Y} = -\frac{ds}{2y^{\frac{3}{2}}}, \qquad \text{Y}' = \frac{dy}{ds\sqrt{y}},$$

$$\text{Z} = 0, \qquad \text{Z}' = \frac{dz}{ds\sqrt{y}}.$$

The first and third of the equations [2] become

$$d\left(\frac{dx}{ds\sqrt{y}}\right) = 0, \quad d\left(\frac{dz}{ds\sqrt{y}}\right) = 0,$$

$$\therefore dx = c\sqrt{y}ds, \qquad dz = c'\sqrt{y}ds.$$

Eliminating ds by these, we obtain $c'dx = cdz$, which being integrated gives $c'x = cz$, no constant being necessary, since the curve passes through the origin of co-ordinates. This being the equation of a right line, shows that the curve sought is a curve described in the vertical plane through the two given points. If this plane be assumed as that of the axes of x and y, the integral of the equation

$$dx = c\sqrt{y}ds$$

will give the sought curve. Let it be squared, and the value of $ds^2 = dy^2 + dx^2$ substituted, and we find

$$dx^2(1 - c^2y) = c^2ydy^2,$$

$$\therefore dx = \frac{c\sqrt{y}dy}{\sqrt{1-c^2y}},$$

which is the differential equation of a *cycloid*. This curve is therefore the *brachystochrone*.

To complete the solution, it is only necessary to restrict the cycloid to pass through the two given points.

Let a be the axis of the cycloid. By comparing the differential equation just found with that of the cycloid, we find

$$a = \frac{1}{c^2}.$$

By which substitution, the equation becomes

$$dx = \frac{\sqrt{y}dy}{\sqrt{a-y}}.$$

It is evident that the base of the cycloid is horizontal, and its axis vertical. The value of a must be selected, so that it shall pass through the two points.

(489.) If the problem be to find the line of swiftest descent from a fixed point to a given curve, then $L'' = 0$, and

$$L' = \frac{dx'}{\sqrt{y'}ds'}\delta x' + \frac{dy'}{\sqrt{y'}ds'}\delta y'.$$

Let the equation of the limiting curve be $y = f(x)$, and let $\delta x' - p\delta y' = 0$. Multiplying this by the arbitrary coefficient A, and adding it to the former, we find

$$L' = \left(\frac{dx'}{\sqrt{y'}ds'} + A\right)\delta x' + \left(\frac{dy'}{\sqrt{y'}\delta s'} - pA\right)\delta y',$$

$$\because \frac{dx'}{\sqrt{y'}ds'} + A = 0, \quad \frac{dy'}{\sqrt{y'}ds'} - pA = 0.$$

Eliminating A, we obtain

$$\frac{dx'}{dy'} = -\frac{1}{p}.$$

Hence the cycloid must be drawn so as to be perpendicular to the proposed limiting curve at the point where they intersect, or the normal of each must be a tangent to the other.

In the same manner, if the problem were to find the line of swiftest descent from one curve to another, it would be a cycloid intersecting both perpendicularly.

PROP. CXXIII.

(490.) *A solid of revolution moves in a fluid in the direction of its axis, to determine its figure so that,* cæteris paribus, *it will suffer least resistance* *.

By the established principles of Mechanics, the resistance which the solid suffers is represented by the integral

$$\int \frac{ydy^3}{dy^2 + dx^2}.$$

Hence

* Newton, Princip. prop. 34, lib. ii. Scholium.

$$\delta\int\frac{ydy^3}{dy^2+dx^2}$$

$$=\int\left\{\frac{ds^2(dy^3\delta y+3ydy^2\delta dy)-2ydy^3(dy\delta dy+dx\delta dx)}{ds^4}\right\}.$$

Hence we find

$$\text{X}=0,\quad \text{X}'=-\frac{2ydy^3dx}{ds^4},$$

$$\text{Y}=\frac{dy^3}{ds^2},\quad \text{Y}'=\frac{ydy^2(dy^2+3dx^2)}{ds^4}.$$

The first two equations of [2] become

$$d\left(\frac{2ydy^3dx}{ds^4}\right)=0,$$

$$\frac{dy^3}{ds^2}-d\left(\frac{ydy^2(dy^2+3dx^2)}{ds^4}\right)=0.$$

If $\dfrac{dy}{dx}=y'$, it is evident that

$$d\left(\frac{y'^3y}{1+y'^2}\right)=\frac{y'^3}{1+y'^2}dy+\frac{3+y'^2}{(1+y'^2)^2}yy'^2dy'.$$

But

$$\text{Y}=\frac{y'^3dx}{1+y'^2},\quad \text{Y}'=\frac{3+y'^2}{(1+y'^2)^2}yy'^2.$$

Hence

$$d\left(\frac{y'^3y}{1+y'^2}\right)=\text{Y}y'+\text{Y}'dy'.$$

But $\text{Y}=d\text{Y}'$. Therefore, by integrating we obtain

$$c+\frac{y'^3y}{1+y'^2}=\text{Y}'y'=\frac{yy'^3(3+y'^2)}{1+y'^2}$$

$$\therefore c(1+y'^2)=2yy'^3.$$

The same result exactly would be obtained by integrating the former of our equations. We have then the equations

$$y=\frac{c(1+y'^2)}{2y'^3},$$

$$x = \int \frac{dy}{y'} = \frac{y}{y'} + \int \frac{y dy'}{y'^2}.$$

Substituting for y in the last integral its value derived from the first equation, and integrating the result, we shall have two equations free from integral signs involving x, y, and y'. By these, y' being eliminated, we shall have the equation of the sought curve.

PROP. CXXIV.

(491.) *To determine the curve of a given length joining two given points, of which the centre of gravity is lowest.*

In this case,

$$\int ds = const., \qquad \int y ds = max.$$

$$\int \frac{dy\delta dy + dx\delta dx}{ds} = 0,$$

$$\int \left(ds\delta y + y \frac{dy\delta dy + dx\delta dx}{ds} \right) = 0.$$

Multiplying the former by the indeterminate const. A, and adding, we find

$$\int \left(ds\delta y + (y + \text{A}) \frac{dy\delta dy + dx\delta dx}{ds} \right) = 0,$$

$$\therefore \text{X} = 0, \quad \text{X}' = (y + \text{A}) \frac{dx}{ds},$$

$$\therefore (y + \text{A}) \frac{dx}{ds} = c,$$

$$\therefore dx = \frac{c dy}{[(y+\text{A})^2 - c^2]^{\frac{1}{2}}},$$

which is the differential equation of the catenary *.

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* Geometry, note on Art. 652, 653.

PART IV.

THE CALCULUS OF DIFFERENCES.

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SECTION I.

Definitions and Notation.

(492.) If the numbers of the arithmetical series

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \cdot\ \cdot\ \cdot\ \cdot$$

be successively substituted for the variable in any function, that function will assume a series of corresponding values, which will, in general, depend on the form of the function, the values of its constants, and the particular number of the series which is substituted for the variable.

Although the differences between every pair of successive values of the variable are equal, being unity, yet it is obvious that this will not, in general, be the case with the differences between the pairs of corresponding values of the function. The value of any such difference will depend on the values which have been assigned to the variable.

If u be taken to express the form of a function of x, the value which u assumes when 0 is substituted for x is expressed thus, u_0; and, in general, the several values of the function u corresponding to the values

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \cdot\ \cdot\ \cdot\ \cdot\ x$$

of the variable are expressed thus,

$$u_0, u_1, u_2, u_3, u_4, u_5, \dots u_x.$$

These are sometimes also expressed thus,

$$^0u, {}^1u, {}^2u, {}^3u, {}^4u, {}^5u, \dots {}^xu.$$

The series of negative integers

$$-1, -2, -3, \dots -x,$$

may also be substituted for the variable, and the results expressed thus,

$$u_{-1}, u_{-2}, u_{-3}, \dots u_{-x},$$

$$\text{or } {}^{-1}u, {}^{-2}u, {}^{-3}u, \dots {}^{-x}u.$$

(493.) Every series, the terms of which increase or decrease by any fixed law or condition, may be conceived to be *generated* by this successive substitution for the variable in a function, which function is called the *general term*, and expresses by its form the *law of the series*. The variable x is called the *index* of the term in which it occurs.

Thus, for example, let the series be

$$a, a + b, a + 2b, a + 3b, \dots a + xb.$$

In this case the general term is

$$u_x = a + xb.$$

By successively substituting

$$0, 1, 2, \dots$$

for x in u_x, the successive terms may be found.

If the series do not commence at a, the preceding terms may be found by substituting successively

$$-1, -2, -3, \dots$$

for x.

Thus it appears, that the nature or law of an arithmetic series is expressed by the equation

$$u_x = a + xb.$$

Again, if the series be

$$a, ar, ar^2, ar^3, \dots$$

The general term is

$$u_x = ar^x,$$

in which the successive substitution of

$$0, 1, 2, 3, \ldots$$

for x, gives a and the terms which succeed it; and the substitution of

$$-1, -2, -3, \ldots$$

gives the terms which precede it.

As in geometry, lines are always supposed to be extended indefinitely in both directions, unless the contrary be expressed; so in the calculus of differences, series are supposed to be continued through an infinite number of terms, unless the question imposes express limits upon them, or they assume limits from the nature of their general term or generatrix.

(494.) The difference between two values of the function which correspond to two successive values of the variable is called the *difference of the function.* The notation expressing this difference should also express the value of the variable in one of the two states of the function. If then the two successive values of the variable be 1 and 2, the corresponding values of the function are

$$u_1, \quad u_2,$$

and the difference

$$u_2 - u_1,$$

which is usually expressed thus, Δu_1.

In general, if the two successive values of the variable be x and $x + 1$, those of the function are

$$u_x, \quad u_{x+1};$$

and the difference is

$$\Delta u_x.$$

The several differences

$$u_1 - u_0, \; u_2 - u_1, \; u_3 - u_2, \ldots$$

are therefore expressed,

$$\Delta u_0, \quad \Delta u_1, \quad \Delta u_2, \; . \; . \; . \; .$$

(495.) It is obvious that the difference Δu_x of a function is itself a function of the variable, and receives a succession of different values by the substitution of the successive integers for the variable. It therefore has a *difference* in the same sense as the function itself. The *difference* of the difference of a function Δu_x would be therefore expressed thus,

$$\Delta(\Delta u_x),$$

or more simply,

$$\Delta^2 u_x.$$

This being again a function of the variable, we find by continuing the same reasoning, a series of successive differences,

$$\Delta u_x, \; \Delta^2 u_x, \; \Delta^3 u_x, \; \Delta^4 u_x, \; . \; . \; . \; .$$

and, in general, $\Delta^n u_x$, which are called the *first difference*, the *second difference*, &c. and, in general, the *n*th *difference*.

The analogy of this language and notation to those of the differential calculus is sufficiently obvious.

(496.) The calculus of differences may be divided into two parts analogous to those of the differential and integral calculus. The *direct calculus of differences*, the object of which is the determination of the successive differences when the function is given; and the *inverse calculus of differences*, the object of which is the determination of the function when the difference is given.

SECTION II.

Of the direct method of differences.

PROP. CXXV.

(497.) *To determine the difference of the algebraical sum of several functions of the same variable.*

Let

$$u_x = u'_x + u''_x - u'''_x,$$
$$\therefore u_{x+1} = u'_{x+1} + u''_{x+1} - u'''_{x+1}.$$

Subtracting, we find

$$\Delta u_x = \Delta u'_x + \Delta u''_x - \Delta u'''_x.$$

And, in general, if

$$u_x = \Sigma(u'_x),$$
$$\therefore \Delta u_x = \Sigma(\Delta u'_x).$$

PROP. CXXVI.

(498.) *The constant quantities connected with the variable of a function by addition or subtraction disappear in its difference; and those united by multiplication or division are united in the same manner with its difference.*

1°. Let the function be

$$u_x + a.$$

Hence the difference is

$$\Delta(u_x + a) = (u_{x+1} + a) - (u_x + a) = u_{x+1} - u_x = \Delta u_x,$$
$$\therefore \Delta(u_x + a) = \Delta u_x.$$

2°. Let the function be

$$au_x,$$

$$\therefore \Delta(au_x) = au_{x+1} - au_x = a(u_{x+1} - u_x).$$

But

$$\Delta u_x = u_{x+1} - u_x,$$
$$\therefore \Delta(au_x) = a\Delta u_x.$$

PROP. CXXVII.

(499.) *To determine the values of* u_x *and* Δu_x *in a series of* u_0, *and its successive differences.*

By what has been already explained, we have

$$u_1 = u_0 + \Delta u_0,$$
$$\therefore \Delta u_1 = \Delta u_0 + \Delta^2 u_0,$$
$$\therefore u_1 + \Delta u_1 = u_0 + 2\Delta u_0 + \Delta^2 u_0.$$

Also,

$$u_2 = u_1 + \Delta u_1$$
$$\therefore u_2 = u_0 + 2\Delta u_0 + \Delta^2 u_0,$$
$$\therefore \Delta u_2 = \Delta u_0 + 2\Delta^2 u_0 + \Delta^3 u_0,$$

which, by addition, gives

$$u_2 + \Delta u_2 = u_0 + 3\Delta u_0 + 3\Delta^2 u_0 + \Delta^3 u_0.$$

But, also,

$$u_3 = u_2 + \Delta u_2,$$
$$\therefore u_3 = u_0 + 3\Delta u_0 + 3\Delta^2 u_0 + \Delta^3 u_0,$$
$$\therefore \Delta u_3 = \Delta u_0 + 3\Delta^2 u_0 + 3\Delta^3 u_0 + \Delta^4 u_0,$$

which, by addition and a similar substitution, gives

$$u_4 = u_0 + 4\Delta u_0 + 6\Delta^2 u_0 + 4\Delta^3 u_0 + \Delta^4 u_0,$$
$$\therefore \Delta u_4 = \Delta u_0 + 4\Delta^2 u_0 + 6\Delta^3 u_0 + 4\Delta^4 u_0 + \Delta^5 u_0;$$

and, in general,

$$u_x = u_0 + \frac{x}{1}\Delta u_0 + \frac{x.x-1}{1.2}\Delta^2 u_0 + \frac{x.x-1.x-2}{1.2.3}\Delta^3 u_0 \cdots [\text{A}],$$

$$\Delta u_x = \Delta u_0 + \frac{x}{1}\Delta^2 u_0 + \frac{x.x-1}{1.2}\Delta^3 u_0 + \frac{x.x-1.x-2}{1.2.3}\Delta^4 u_0 \cdots [\text{B}],$$

and in like manner we have, in general,

$$u_{x+n}=u_n+\frac{x}{1}\Delta u_n+\frac{x.x-1}{1.2}\Delta^2 u_n+\frac{x.x-1.x-2}{1.2.3}\Delta^3 u_n\ldots[\text{C}];$$

or,

$$u_{n+x}=u_x+\frac{n}{1}\Delta u_x+\frac{n.n-1}{1.2}\Delta^2 u_x+\frac{n.n-1.n-2}{1.2.3}\Delta^3 u_x\ldots[\text{D}].$$

This series may easily be retained in the memory, by observing that it is the development of

$$u_x(1+\Delta)^n;$$

and the former shows that u_{x+n} may be expressed thus,

$$u_n(1+\Delta)^x.$$

So that any of the four expressions may be indiscriminately used one for the other.

PROP. CXXVIII.

(500.) *To determine* $\Delta^n u_x$ *in a series of* u_{x+n}, u_{x+n-1}, u_{x+n-2}. . . .

By what has been established, we have

$$\Delta u_0 = u_1 - u_0,$$
$$\Delta^2 u_0 = \Delta u_1 - \Delta u_0;$$

but, also,

$$\Delta u_1 = u_2 - u_1,$$
$$\therefore \Delta^2 u_0 = u_2 - 2u_1 + u_0.$$

By taking the difference of this, we find

$$\Delta^3 u_0 = \Delta u_2 - 2\Delta u_1 + \Delta u_0.$$

Substituting for these differences their values, we have

$$\Delta^3 u_0 = u_3 - 3u_2 + 3u_1 - u_0.$$

In like manner, taking the differences of these,

$$\Delta^4 u_0 = \Delta u_3 - 3\Delta u_2 + 3\Delta u_1 - \Delta u_0.$$

Substituting as before, we find

$$\Delta^4 u_0 = u_4 - 4u_3 + 6u_2 - 4u_1 + u_0,$$

$$\therefore \Delta^n u_0 = u_n - \frac{n}{1}u_{n-1} + \frac{n.n-1}{1.2}u_{n-2} - \frac{n.n-1.n-2}{1.2.3}u_{n-3}\ldots$$

and, in general,

$$\Delta^n u_x = u_{x+n} - \frac{n}{1} u_{x+n-1} + \frac{n.n-1}{1.2} u_{x+n-2} - \frac{n.n-1.n-2}{1.2.3} u_{x+n-3}.$$

Thus the value of $\Delta^n u_x$ is equivalent to the development of

$$(u - 1)^{x+n}.$$

The exponents of u in the successive terms being removed below the letter thus, u_{x+n}.

(501.) When the function is given, its successive differences are easily obtained.

Let $u_n = (y + nh)^m$, $\because$

$$u_0 = y^m,$$
$$u_1 = (y + h)^m,$$
$$u_2 = (y + 2h)^m,$$
$$u_3 = (y + 3h)^m,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence

$$\Delta u_0 = (y + h)^m - y^m,$$

$$\because \Delta u_0 = my^{m-1}h + \frac{m.m-1}{1.2} y^{m-2}h^2 + \frac{m.m-1.m-2}{1.2.3} y^{m-3}h^3 + \cdots$$

To obtain the second, third, and succeeding differences, it is necessary to change y into $y + h$ in Δu, $\Delta^2 u$. . . . . .
Hence we obtain

$$\Delta u_1 = m(y + h)^{m-1}h + \frac{m.m-1}{1.2}(y + h)^{m-2}h^2 + \cdots$$

It is evident that by developing Δu_1, and arranging the result by the ascending powers of h, and subtracting Δu from the result, the series will have the form

$$\Delta^2 u = m.m - 1 \,.\, y^{m-2}h^2 + M_3 y^{m-3}h^3 + M_4 x^{m-4}h^4 + \cdots$$

where M_3, $M_4 \cdots$ signify the functions of m, which form the successive coefficients.

By a similar substitution in this last series, and observing

the condition $\Delta^3 u = \Delta^2 u_1 - \Delta^2 u,$ we shall obtain

$$\Delta^3 u = m.m - 1.m - 2.y^{m-3}h^3 + M'_4 y^{m-4}h^4 + \cdots\cdot$$

It is obvious then that the first term of the development of $\Delta^n u$ must be

$$m\cdot m - 1\cdot m - 2\cdots\cdot(m - n + 1)y^{m-n}h^n.$$

It follows, therefore, that when the exponent m is a positive integer, the number of terms of the development of $\Delta^n u$, arranged by the powers of y, diminishes by unity as n increases by unity, and that when $n = m$, we have

$$\Delta^m n = m\cdot m - 1\cdot m - 2\cdots\cdot 3.2.1.h^m.$$

This difference being constant, all the succeeding differences must $= 0$.

We can obtain the general term of the series $\Delta^n u$ by means of the values of u, u_1, u_2, $\cdots\cdot$ independently of Δu, $\Delta^2 u$, $\Delta^3 u$, $\cdots\cdot$ We have

$$u_1 = (y + h)^m,$$
$$u_2 = (y + 2h)^m,$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$u_n = (y + nh)^m.$$

Hence by (500.)

$$\Delta^n u = (y+nh)^m - \frac{n}{1}[y+(n-1)h]^m + \frac{n.n-1}{1.2}[y+(n-2)h]^m$$
$$- \frac{n.n-1.n-2}{1.2.3}[y + (n-3)h]^m + \cdots\cdot$$

If i be the exponent of h in any term of this, when each term shall have been separately developed, the general expression for this term will be

$$\frac{m.m-1.m-2\cdots\cdot(m-i+1)}{1.2.3\cdots\cdots\cdots i}y^{m-i}h^i \times$$
$$\left\{ n^i - \frac{n}{1}(n-1)^i + \frac{n.n-1}{1.2}(n-2)^i - \cdots \right\}.$$

But the development of $\Delta^n u$ cannot involve any powers of h of which the exponents are less than n, as appears from the lowest exponent in the development of $\Delta^n u$ being

n. Hence it follows, that the function

$$n^i - \frac{n}{1}(n-1)^i + \frac{n.n-1}{1.2}(n-2)^i - \cdots$$

consisting of $(n+1)$ terms must $= 0$ when $i < n$.

Also the coefficient

$$\frac{m.m-1.m-2 \cdots (m-i+1)}{1.2.3 \cdots\cdots i}$$

must vanish when $m = i + 1$. It follows, therefore, that no power of h in the development of $\Delta^n u$ can have a higher exponent than m.

PROP. CXXIX.

(502.) *To determine the successive differences of a rational and integral function of* x.

The form of the proposed function is

$$u = \text{A}x^a + \text{B}x^b + \text{C}x^c + \text{D}x^d \cdots$$

Taking the nth difference

$$\Delta^n u = \text{A}\Delta^n x^a + \text{B}\Delta^n x^b + \text{C}\Delta^n x^d \cdots *$$

by (498.).

The nth differences of each of the powers of x must then be separately found by the methods given in (501.).

If a be the highest exponent in the series, we have

$$\Delta^a x^a = 1.2.3 \cdots ah^a,$$

$$\Delta^a x^b = 0, \quad \Delta^a x^c = 0 \cdots$$

Hence

$$\Delta^a u = 1.2.3 \cdots a\text{A}h^a.$$

---

* By $\Delta^n x^a$ I denote the nth difference of x^a; and $(\Delta^n x)^a$ expresses the ath power of the nth difference of x. Lacroix expresses the former by $\Delta^n . x^a$, and the latter $\Delta^n x^a$. I do not think these sufficiently distinct.

(503.) Hence it follows, that "every rational and integral function of x has a constant difference, and the order of this difference is expressed by the exponent of the highest power of x which enters the function."

(504.) Hence every function which admits of being expanded in a finite series of ascending integral and positive powers of x, has a constant difference of the nth order, n being the exponent of x in the last term.

(505.) No function of x whose development in ascending positive and integral powers of x is not finite, can have a constant difference of any order.

(506.) The following example will illustrate what has just been established.

Let

$$u_x = x^3 + 2x + 3.$$

Substituting successively 0, 1, 2, 3, ... for x, we obtain the values of $u_0, u_1, u_2, u_3, \cdots$ By subtraction, we thence obtain the values of $\Delta u_0, \Delta u_1, \Delta u_2, \Delta u_3, \cdots$ and thence, in like manner, the values of $\Delta^2 u_0, \Delta^2 u_2, \Delta^2 u_3, \cdots$ and so on. Thus,

$$u_0 = 3, \quad u_1 = 6, \quad u_2 = 15, \quad u_3 = 36, \quad u_4 = 75 \cdots\cdots$$

$$\Delta u_0 = 3, \quad \Delta u_1 = 9, \quad \Delta u_2 = 21, \quad \Delta u_3 = 39 \cdots\cdots$$

$$\Delta^2 u_0 = 6, \quad \Delta^2 u_1 = 12, \quad \Delta^2 u_2 = 18 \ldots\ldots$$

$$\Delta^3 u_0 = 6, \quad \Delta^3 u_1 = 6 \ldots\ldots$$

$$\Delta^4 u_0 = 0 \ldots\ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Here we perceive that the differences of the second order are in arithmetical progression, those of the third order equal, and all superior orders = 0.

It will easily appear that this is universally the case with rational and integral functions.

(507.) The calculus of differences is of considerable use in approximating to the values of transcendental functions,

as in the calculation of trigonometrical, logarithmic, and other tables.

Let $u_0 = lx$. Hence

$$u_1 = l(x+h),$$
$$u_2 = l(x+2h),$$
$$u_3 = l(x+3h),$$
$$\therefore \Delta u_0 = l(x+h) - lx,$$
$$= \text{M}\left\{\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \dots\right\},$$
$$\Delta^2 u_0 = l(x+2h) - 2l(x+h) + lx,$$
$$= l\left(1+\frac{2h}{x}\right) - 2l\left(1+\frac{h}{x}\right),$$
$$= -\text{M}\left\{\frac{h^2}{x^2} - \frac{2h^3}{x^3} + \dots\right\},$$
$$\Delta^3 u_0 = l(x+3h) - 3l(x+2h) + 3l(x+h) - lx,$$
$$= l\left(1+\frac{3h}{x}\right) - 3l\left(1+\frac{2h}{x}\right) + 3l\left(1+\frac{h}{x}\right),$$
$$= \text{M}\left\{\frac{2h^3}{x^3} - \dots\right\}.$$

These differences must be continued until one is found so small, that it may be neglected without producing an error of any practical importance in the proposed calculation.

Suppose, for example, that $x = 10000$ and $h = 1$, we should then have

$$u = l\,10000$$
$$\Delta u = 0{,}00004\ 34272\ 76863$$
$$\Delta^2 u = -0{,}0000\ 00043\ 42076$$
$$\Delta^3 u = 0{,}0000\ 00000\ 00868.$$

If in the final result it should be only required to proceed as far as ten places, the differences of the fourth order might be neglected for the several successive numbers, for they should be repeated very often before they could pro-

duce any effect upon the difference of the third order. If the differences of the third, second, and first orders be found, the logarithms of 10001, 10002, 10003, ... may be computed when that of 10000 is known, which is

$$4.00000\ 00000\ 00000.$$

It is necessary that the calculation should be extended to fifteen decimal places, in order to determine when the accumulation of error arising from the value of the neglected places will begin to produce an effect upon the last figure, to which it is proposed to extend the computed result. This may be always determined by the logarithms of numbers rigorously computed, *a priori*, at stated intervals, and by comparison with which it may be ascertained. If the first ten places be not the same, the difference has been taken as constant through too great a number of terms.

The formula [A] determined in (499.),

$$u_n = u_0 + \frac{n}{1}\Delta u_0 + \frac{n.n-1}{1.2}\Delta^2 u_0 + \frac{n.n-1.n-2}{1.2.3}\Delta^3 u_0 + \ldots$$

furnishes an easy method of determining the error produced by the suppression of the differences of any proposed order.

In the example already given, let $n = 50$, and let the corresponding value of

$$\frac{n.n-1.n-2.n-3}{1.2.3.4}\Delta^4 u_0$$

be computed. Since

$$\Delta^4 u_0 = -\text{M}\left(\frac{6n^4}{x^4} - \ldots.\right),$$

we find that it produces no influence upon the tenth decimal place of the logarithm of 10050. It will be therefore *a fortiori* the same for differences of superior orders.

SECTION III.

Of interpolation.

(508.) In calculating the various tables used in the different departments of physical science, the process would be elaborate in the extreme, if each particular number required a separate, *a priori*, computation. To remedy this inconvenience, mathematicians propose the following problem:

> "Given several terms of a series to introduce between them other terms in such a manner that the *law of the series* shall not be changed."

The solution of this problem is called *the method of interpolation.*

If the law of the series were explicitly given, the solution would be obvious. For, by this law, the general term would be expressed, and the successive substitution of any series of proposed values for the variable in that term would introduce the required terms in the series. This must be obvious from what has been observed in Section I.

In this point of view the problem is equivalent to being given any number of ordinates of a given curve to draw the intermediate ordinates, which correspond to any proposed intermediate abscissæ, which can always be done when the equation of the curve is given. The value of the ordinate derived from the equation of the curve is, in this case, the general term of the series.

The case in which the use of the method of interpolation is more generally required, is that in which the law of the series is not given; but only the numerical values of cer-

tain terms of the series at stated intervals. The law in this case cannot be known, but may in a manner be approximated to.

In this case the question becomes equivalent to drawing a curve through a number of given points, *the species of the curve not being given*, which is a question evidently indeterminate. The calculus of differences, however, presents a method of solving the problem approximately.

(509.) Let u be the general term of the series, and let

$$u_0, u_1, u_2, u_3, \dots$$

be the values of u, which correspond to the particular values,

$$x_0, x_1, x_2, x_3, \dots$$

of the variable. We shall suppose u in general developed in a series of ascending powers of x,

$$u = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3 \dots$$

The several coefficients A, B, C, D, ... may be determined by the supposition that u becomes

$$u_0, u_1, u_2, \dots$$

when x becomes

$$x_0, x_1, x_2, \dots$$

We shall first suppose that this series of values of x are in arithmetical progression.

If x be assumed very small, the series

$$u = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3 \dots$$

may be supposed to end at such a term $\text{M}x^m$ as will leave the remainder so small as to produce an inconsiderable effect upon the value of u.

The mth difference of the function u may, under these circumstances, be considered constant (503.), and, consequently, (499.), we have

$$u_n = u_0 + \frac{n}{1}\Delta u_0 + \frac{n.n-1}{1.2}\Delta^2 u_0 + \frac{n.n-1\,n-2}{1.2.3}\Delta^3 u_0 \dots$$
$$\dots + \frac{n.n-1 \dots (n-m+1)}{1.2.3.\dots m}\Delta^m u_0.$$

By this series, if u_0, Δu_0, $\Delta^2 u_0$, ... be known, the value of u_n will be determined for any value of n.

Let $x = x_0 + nh$, $\therefore n = \frac{x - x_0}{h}$, and if $x - x_0 = h'$,

$\therefore n = \frac{h'}{h}$.

Hence the series becomes

$$u_n = u_0 + \frac{h'}{h}\Delta u_0 + \frac{h'(h'-h)}{h.2h}\Delta^2 u_0 + \frac{h'(h'-h)(h'-2h)}{h.2h.3h}\Delta^3 u + \cdots$$

and if $u_n - u_0 = \Delta' u$, $\because$

$$\Delta' u = \frac{h'}{h}\Delta u_0 + \frac{h'(h'-h)}{h.2h}\Delta^2 u_0 + \frac{h'(h'-h)(h'-2h)}{h.2h.3h}\Delta^3 u_0 + \cdots$$

This being a rational function of h', or $(x - x_0)$, or x, of the same degree as

$$u = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3, \cdot$$

it will be the function required, and may be therefore considered as the general term of the series.

(510.) As an example of the application of this process, let the terms of the series which are given be

$$u_0 = 3, \; u_1 = 7, \; u_2 = 19, \; u_3 = 39, \; u_4 = 67 \cdots$$

Hence,

$$u_0 = 3, \; \Delta u = 4, \; \Delta^2 u = 8, \; \Delta^3 u = 0, \; h = 1.$$

The series $\Delta' u$ is in this case reduced to its first two terms, and becomes

$$\Delta' u = 4h' + 4h'(h' - 1) = 4h'^2.$$

Hence we find

$$u_{h'} = 3 + 4h'^2.$$

Thus, if $h' = \frac{5}{2}$, the corresponding term of the series would be 28; and, in like manner, the term of the series corresponding to any other exponent might be determined.

Again, let the given terms of the series be

$$u_0 = 1, \quad u_1 = 4, \quad u_2 = 2,$$
$$u_3 = 3, \quad u_4 = 9, \quad u_5 = 16.$$

. . . .

Hence

$$u_0 = 1, \quad \Delta u_0 = 3, \quad \Delta^2 u_0 = -5, \quad \Delta^3 u_0 = 8,$$
$$\Delta^4 u = -6, \quad \Delta^5 u = 0, \quad h = 1.$$

Hence

$$u_n = 1 + 3\frac{h'}{h} - 5\frac{h'(h'-1)}{1.2} + 8\frac{h'(h'-1)(h'-2)}{1.2.3}$$
$$- 6\frac{h'(h'-1)(h'-2)(h'-3)}{1.2.3.4},$$

which being developed and arranged by the powers of h', becomes

$$u_n = \frac{12+116h'-111h'^2+34h'^3-3h'^4}{12}.$$

In these cases the law of the series has been rigorously determined, and the values of any proposed term u_n can be determined, not approximately, but exactly. This is always the case when we obtain a constant difference, however high its order may be, because, in that case, the successive values can only result from an algebraic function.

(511.) The series expressing $\Delta' u$ is generally used when the differences Δu_0, $\Delta^2 u_0$, $\Delta^3 u_0$, ... continually decrease, because, in that case, it is convergent. In case the general term of the series be not an algebraic function, the terms intermediate between any two may be determined approximately by assuming one of the differences Δu_0, $\Delta^2 u_0$, &c. of a sufficiently high order, and considering it as constant for all the intermediate terms, and determining the intermediate terms and their differences by the method already given for the case of algebraic functions.

As an example, let it be required to compute the common logarithm of the number

3,1415926536

by means of a table containing the logarithms of all integers from 1 to 1000 to ten decimal places. We shall take these logarithms as particular values of u, and the numbers them-

selves as the indices of the functions; thus, let

$$u_0 = l(3{,}14), \quad u_1 = l(3{,}15), \quad u_2 = l(3{,}16),$$
$$u_3 = l(3{,}17), \quad u_4 = l(3{,}18).$$

Hence by the tables, we have

$$\begin{aligned} u_0 &= 0{,}4969296481 \\ u_1 &= 0{,}4983105538 \\ u_2 &= 0{,}4996870826 \\ u_3 &= 0{,}5010592622 \\ u_4 &= 0{,}5024271200. \end{aligned}$$

Hence we find

$$\begin{aligned} \Delta u_0 &= 0{,}0013809057 \\ \Delta u_1 &= 0{,}0013765288 \\ \Delta u_2 &= 0{,}0013721796 \\ \Delta u_3 &= 0{,}0013678578. \end{aligned}$$

$$\begin{aligned} \Delta^2 u_0 &= -\ 0{,}0000043769 \\ \Delta^2 u_1 &= -\ 0{,}0000043492 \\ \Delta^2 u_2 &= -\ 0{,}0000043218. \end{aligned}$$

$$\begin{aligned} \Delta^3 u_0 &= 0{,}0000000277 \\ \Delta^3 u_1 &= 0{,}0000000274. \end{aligned}$$

$$\Delta^4 u_0 = -\ 0{,}0000000003.$$

By continuing the process, and taking from the tables the logarithms of 3,19, 3,20, &c. we should find the differences $\Delta^5 u_0$, $\Delta^6 u_0$, &c. still decreasing, and for several successive numbers we should find the fourth differences $\Delta^4 u_0$, $\Delta^4 u_1$, $\Delta^4 u_2$, $\Delta^4 u_3 \cdots$ as far as the tenth decimal place, the same as that already found, we assume that in calculating Δu to the tenth place, the series expressing it should rigorously terminate at the fourth term.

Since, then,

$$h = 3{,}15 - 3{,}14 = 0{,}01,$$
$$h' = 3{,}1415926536 - 3{,}14 = 0{,}0015926536.$$

Hence,

$$\frac{h'}{h} = 0{,}15926536$$

$$\frac{h'-h}{2h} = -\ 0{,}42036732,$$

$$\frac{h'-2h}{3h} = -\ 0{,}61357821,$$

$$\frac{h'-3h}{4h} = -\ 0{,}71018366.$$

Substituting these in the formula

$$\Delta' u = \frac{h'}{h}\Delta u_0 + \frac{h'(h'-h)}{h.2h}\Delta^2 u_0 + \frac{h'(h'-h)(h'-2h)}{h.2h.3h}\Delta^3 u_0$$
$$+ \frac{h'(h'-h)(h'-2h)(h'-3h)}{h.2h.3h.4h}\Delta^4 u_0\,;$$

and effecting the operations indicated by the signs, the result is

$$\Delta' u = 0{,}0002202245,$$

$$\therefore \log.\ (3{,}1415926536) = 0{,}4971498726.$$

(512.) In the preceding cases we have supposed that the given values

$$x_0,\quad x_1,\quad x_2,\quad x_3, \cdots\cdots$$

were in arithmetical progression. When this is not the case, let the particular values be successively substituted for x in the series

$$u = \text{A} + \text{B}x + \text{C}x^2 + \text{D}x^3 + \cdots\cdots$$

which gives

$$u_0 = \text{A} + \text{B}x_0 + \text{C}x_0^2 + \text{D}x_0^3 \cdots\cdots$$
$$u_1 = \text{A} + \text{B}x_1 + \text{C}x_1^2 + \text{D}x_1^3 \cdots\cdots$$
$$u_2 = \text{A} + \text{B}x_2 + \text{C}x_2^2 + \text{D}x_2^3 \cdots\cdots$$
$$u_3 = \text{A} + \text{B}x_3 + \text{C}x_3^2 + \text{D}x_3^3 \cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

The number of given values u_0, u_1, $u_2 \cdots\cdots$ ought to be equal at least to the number of coefficients A, B, C, $\cdots\cdots$ which it is required to determine.

By subtracting successively each equation from that which follows it, and dividing the successive results by $x_1 - x_0$, $x_2 - x_1$, ... the results will be

$$\frac{u_1 - u_0}{x_1 - x_0} = B + C(x_1 + x_0) + D(x_1^2 + x_1 x_0 + x_0^2) \ldots\ldots$$

$$\frac{u_2 - u_1}{x_2 - x_1} = B + C(x_2 + x_1) + D(x_2^2 + x_2 x_1 + x_1^2) \ldots\ldots$$

$$\frac{u_3 - u_2}{x_3 - x_2} = B + C(x_3 + x_2) + D(x_3^2 + x_3 x_2 + x_2^2) \ldots\ldots$$

. . . .

. . . .

Let

$$\frac{u_1 - u_0}{x_1 - x_0} = U_0, \quad \frac{u_2 - u_1}{x_2 - x_1} = U_1, \text{ \&c.}$$

Subtracting U_0 from U_1, U_1 from U_2, &c. and dividing the successive results by $x_2 - x_1$, $x_3 - x_2$, &c. and calling the quantities

$$\frac{U_1 - U_0}{x_2 - x_1}, \quad \frac{U_2 - U_1}{x_3 - x_1}, \text{ \&c.}$$

U'_0, U'_1, &c. we obtain

$$U'_0 = C + D(x_2 + x_1 + x_0) \ldots\ldots$$
$$U'_1 = C + D(x_3 + x_2 + x_1) \ldots\ldots$$

from whence we find

$$U'_1 - U'_0 = D(x_3 - x_0).$$

Substituting U'' for

$$\frac{U'_1 - U'_0}{x_3 - x_0},$$

we have $U''_0 = D +$, &c. If we suppose that the first four terms give a sufficient approximation to u, we shall have

$$D = U''_0,$$
$$C = U'_0 - U''_0(x_2 + x_1 + x_0),$$
$$B = U_0 - U'_0(x_1 + x_0) + U''_0(x_2 x_1 + x_2 x_0 + x_1 x_0),$$
$$A = u_0 - U_0 x_0 + U'_0 x_1 x_0 - U'' x_2 x_1 x_0.$$

Substituting these values in the general expression for u, we find

$$u=u_0+\text{U}_0(x-x_0)+\text{U}'_0[x^2-(x_1+x_0)x+x_1x_0]$$
$$+\text{U}''_0[x^3-(x_2+x_1+x_0)x^2+(x_2x_1+x_2x_0+x_1x_0)x-x_2x_1x_0].$$

By this reasoning, we should have a formula similar to this, whatever may be the number of given values of x, and it may in general be expressed thus:

$$u=u_0+\text{U}_0(x-x_0)+\text{U}'_0(x-x_0)(x-x_1)+\text{U}''_0(x-x_0)(x-x_1)(x-x_2)$$
$$+\text{U}'''_0(x-x_0)(x-x_1)(x-x_2)(x-x_3)+\ldots.$$

The meaning of the several coefficients being determined by

$$\frac{u_1-u_0}{x_1-x_0}=\text{U}_0,\ \frac{u_2-u_1}{x_2-x_1}=\text{U}_1,\ \frac{u_3-u_2}{x_3-x_2}=\text{U}_2,\ \frac{u_4-u_3}{x_4-x_3}=\text{U}_3,\ \&c.$$

$$\frac{\text{U}_1-\text{U}_0}{x_2-x_0}=\text{U}'_0,\ \frac{\text{U}_2-\text{U}_1}{x_3-x_1}=\text{U}'_1,\ \frac{\text{U}_3-\text{U}_2}{x_4-x_2}=\text{U}'_2,\ \&c.$$

$$\frac{\text{U}'_1-\text{U}'_0}{x_3-x_0}=\text{U}''_0,\ \frac{\text{U}'_2-\text{U}'_1}{x_4-x_1}=\text{U}''_1,\ \&c.$$

$$\frac{\text{U}''_1-\text{U}''_0}{x_4-x_0}=\text{U}'''_0,\ \&c.$$

(513.) The series already found for the case in which the values

$$x_0,\quad x_1,\quad x_2,\quad x_3,\ \ldots.$$

are in arithmetical progression, may, without difficulty, be deduced from the more general formula which we have just established.

In this case we have

$$x_1-x_0=x_2-x_1=x_3-x_2=\ldots.$$

Hence

$$x_1=x+h,$$
$$x_2=x+2h,$$
$$x_3=x+3h.$$
$$\ldots\ldots$$
$$\ldots\ldots$$

Therefore

$$\text{U}_0 = \frac{\Delta u_0}{h}, \quad \text{U}_1 = \frac{\Delta u_1}{h}, \quad \text{U}_2 = \frac{\Delta u_2}{h}, \text{ \&c.}$$

$$\text{U}'_0 = \frac{\Delta^2 u_0}{1.2h^2}, \quad \text{U}'_1 = \frac{\Delta^2 u_1}{1.2h^2}, \quad \text{U}'_2 = \frac{\Delta^2 u_2}{1.2h}, \text{ \&c.}$$

$$\text{U}''_0 = \frac{\Delta^3 u_0}{1.2.3h^3}, \quad \text{U}''_1 = \frac{\Delta^3 u_1}{1.2.3h^3}, \text{ \&c.}$$

$$\text{U}'''_0 = \frac{\Delta^4 u_0}{1.2.3h^4}, \text{ \&c.}$$

. . . .

Let $x = x_0 + h'$, $\therefore$

$x - x_0 = h'$, $x - x_1 = h' - h$, $x - x_2 = h' - 2h$,

$x - x_3 = h' - 3h$ . . . .

By which the formula found in the preceding article becomes

$$u = u_0 + \frac{h'}{h}\Delta u_0 + \frac{h'(h'-h)}{h.2h}\Delta^2 u_0 + \frac{h'(h'-h)(h'-2h)}{h.2h.3h}\Delta^3 u_0 \dots$$

which is the same as the result of (509.).

(514.) The general formula for u may also be expressed in another way. Since the values of

$$u_0, \ u_1, \ u_2, \dots$$

in terms of

$$x_0, \ x_1, \ x_2, \dots$$

are all simple equations with respect to the several coefficients A, B, C, ...

It follows that the expression for u should be such, that by changing x successively into x_0, x_1, x_2, ... u should become u_0, u_1, u_2, . . . Hence we should have

$$u = \text{X}u_0 + \text{X}_1 u_1 + \text{X}_2 u_2 + \dots$$

provided that the functions X_0, X_1, X_2, ... be such, that the successive substitutions of x_0, x_1, x_2, ... for x give

$$\text{X}_0 = 1, \quad \text{X}_1 = 0, \quad \text{X}_2 = 0, \quad \text{X}_3 = 0, \dots$$

$$\text{X}_0 = 0, \quad \text{X}_1 = x_1, \quad \text{X}_2 = 0, \quad \text{X}_3 = 0, \dots$$

$$\text{X}_0 = 0, \quad \text{X}_1 = 0, \quad \text{X}_2 = x_2, \quad \text{X}_3 = 0, \dots$$

. . . .

which conditions are satisfied by

$$\mathrm{X}_0 = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots\ldots}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots\ldots},$$

$$\mathrm{X}_1 = \frac{(x-x_0)(x-x_2)(x-x_3)\ldots\ldots}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots\ldots},$$

$$\mathrm{X}_2 = \frac{(x-x_0)(x-x_1)(x-x_3)\ldots\ldots}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots\ldots}.$$

$$\cdots\cdots\cdots\cdots\cdots$$

The numerator and denominator of these several expressions each contains a number of factors one less than the number of given values of x. The formula for interpolation therefore becomes

$$u = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots\ldots}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots\ldots}u_0,$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots\ldots}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots\ldots}u_1,$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)\ldots\ldots}{(x_2-x_1)(x_2-x_1)(x_2-x_3)\ldots\ldots}u_2.$$

$$+\cdots\cdots\cdots\cdots\cdots$$

This formula is particularly adapted for computation, since each term may be calculated by logarithms. See Geometry (617.).

(515.) The method of quadratures, or of approximating to the value of the integral $\int \mathrm{x}\,dx$, is facilitated by interpolation.

Let the curve, of which the ordinate is $u = \mathrm{x}$, and of which the area is therefore $\int \mathrm{x}\,dx$, be supposed to be intersected in a certain number of given points by a parabolic curve represented by the equation

$$u = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 \ldots\ldots$$

the coefficients being indeterminate.

The area of this curve will be

$$\int u\,dx = \mathrm{A}\frac{x}{1} + \mathrm{B}\frac{x^2}{2} + \mathrm{C}\frac{x^3}{3} + \mathrm{D}\frac{x^4}{4}\ldots\ldots + \text{const.}$$

By (499.) we have

$$u_x = u_0 + \frac{x}{1}\Delta u_0 + \frac{x.x-1}{1.2}\Delta^2 u_0 + \frac{x.x-1.x-2}{1.2.3}\Delta^3 u_0 \dots$$

each of these series being continued through as many terms as there are points common to the two curves.

Let the number of common points be three. Taking the first three terms of the preceding formulæ, we have

$$\text{A} = u_0, \quad \text{B} = \Delta u_0 - \tfrac{1}{2}\Delta^2 u_0, \quad \text{C} = \tfrac{1}{2}\Delta^2 u.$$

These quantities depend only on the three successive values u_0, u_1, u_2, which correspond to

$$h' = 0,\ h' = h,\ h' = 2x, \text{ or } x = 0,\ x = 1,\ x = 2.$$

If the integral be taken between the limits of the first and last, its value will be

$$2u_0 + 2(\Delta u_0 - \tfrac{1}{2}\Delta^2 u_0) + \tfrac{4}{3}\Delta^2 u_0$$
$$= 2(u_0 + \Delta u_0 + \tfrac{1}{6}\Delta^2 u_0).$$

The value of the integral thus found is the area of the segment of a parabola meeting the proposed curve in three points, and comprised between the ordinates through the first and third point.

It is evident that this parabolic area has a part in common with the area of the proposed curve; and that the second ordinate divides both areas into two parts, one of the parts of the parabolic area exceeding the corresponding part of the required area, and the other falling short of it, the difference of these differences being the error in the total result.

SECTION IV.

The inverse calculus of differences.

(516.) The object of the inverse method of differences is to determine the primitive function from its differences. Thus, as has been already observed, this part has the same relation to the direct calculus of differences as the integral has to the differential calculus.

We shall here confine ourselves to the integration of that class of differences only which are expressed as immediate functions of the independent variable. All such come under the form

$$\Delta^r u_x = \text{F}(x),$$

the increment h of x we shall suppose given and constant.

(517.) There are three theorems which are obvious from the inversion of the corresponding ones in the direct calculus of differences.

1°. That as constants united to any function by addition or subtraction disappear in its difference, so in integrating the difference of a function, an arbitrary constant should be added. Thus,

$$\Sigma(\Delta u_x) = u_x + c.$$

2°. As constants connected by multiplication or division with a function are similarly connected with its difference, so, in integrating, the constants thus connected with the difference should be preserved in its integral. Thus, since

$$\Delta(\text{A}x) = \text{A}\Delta x,$$

$$\therefore \Sigma(\text{A}x) = \text{A}\Sigma x.$$

It should be observed, that the sign Σ before any quan-

tity implies the integral of which that quantity is the difference. Thus, Σx is the integral of which x is the difference. So that

$$\Delta\Sigma x = x, \text{ or } \Sigma\Delta x = x;$$

the operations indicated by Σ and Δ being subversive each of the other.

3°. That as the difference of several quantities united by addition or subtraction is found by uniting similarly their several differences, so also the integral of several differences thus united is found by uniting similarly their several integrals. Thus, since

$$\Delta(x + y - z) = \Delta x + \Delta y - \Delta z,$$
$$\therefore \Sigma(x + y - z) = \Sigma x + \Sigma y - \Sigma z.$$

(518.) When the proposed difference is a rational and integral function of the independent variable, its exact integral may always be determined. It appears from what has been already established, that there is a certain order of differences of such a function which are constant. Let the exponent of this order be m. Since, in general,

$$\Delta^r u_x = \mathrm{F}(x),$$
$$\therefore \Delta^{r+m} u_x = \Delta^m \mathrm{F}(x).$$

Since this latter is constant, we have

$$u_n = u + \frac{n}{1}\Delta u + \frac{n.n-1}{1.2}\Delta^2 u \ldots .$$
$$\ldots . + \frac{n.n-1 \ldots .(n-r-m+1)}{1.2.3 \ldots .(r+m)}\Delta^{r+m} u,$$

in which $u, \Delta u, \Delta^2 u, \ldots .$ are those values which correspond to $x = a$. If $a + nh = x$, $\therefore u_n$ becomes u_x.

If we suppose $v_x = \mathrm{F}(x)$, $\therefore$

$$\Delta^r u = v, \quad \Delta^{r+1} u = \Delta v \ldots . \Delta^{r+m} u = \Delta^m v,$$

u and its differences, as far as the $(r - 1)$th order inclusive, being arbitrary.

As an example of this, let

$$\Delta u_x = x^3 - 5x^2 + 6x - 1,$$

the increment of x being unity. In this case $r = 1$, $m = 3$, $h = 1$. If we suppose $a = 0$, we have

$$v = -1, \quad \Delta v = 2, \quad \Delta^2 v = -4,$$
$$\Delta^3 v = 6, \quad \Delta^4 v = 0.$$

Hence

$$\Sigma(x^3 - 5x^2 + 6x - 1) = u_x =$$
$$u - 1\frac{x}{1} + 2\frac{x(x-1)}{1.2} - 4\frac{x(x-1)(x-2)}{1.2.3}$$
$$+ 6\frac{x(x-1)(x-2)(x-3)}{1.2.3.4} = \frac{3x^4 - 26x^3 + 69x^2 - 58x}{12} + \text{cons.}$$

(519.) The method of integrating an extensive class of differences may be derived from the form of the difference of the formula

$$u = x(x+h)(x+2h) \ldots [x + (m-1)h].$$

The difference of this is

$$\Delta u = (x+h)(x+2h)(x+3h) \ldots (x+mh)$$
$$- x(x+h)(x+2h) \ldots [x+(m-1)h]$$
$$= (x+h)(x+2h) \ldots [x+(m-1)h]mh.$$

Hence by taking the integrals, observing that mh is constant, we obtain

$$\Sigma(x+h)(x+2h)(x+3h) \ldots [x+(m-1)h]$$
$$= \frac{x(x+h)(x+2h) \ldots [x+(m-1)h]}{mh} + \text{C}.$$

By changing x into $x - h$, and m into $m + 1$, this becomes

$$\Sigma x(x+h)(x+2h) \ldots [x+(m-1)h]$$
$$= \frac{(x-h)x(x+h)(x+2h) \ldots [x+(m-1)h]}{(m+1)h} \quad \ldots \quad [1].$$

By means of this formula, every function which can be reduced to a product of equidifferent factors may be integrated. The analogy which the formula just found bears to

$$\int x^m dx = \frac{x^{m+1}}{m+1}$$

is obvious. In both, the number of factors in the numerator is increased by one by the integration, and the factor $m+1$ is introduced into the denominator.

(520.) A method of integrating another class of differences may be deduced from the difference of

$$u = \frac{1}{x(x+h)(x+2h)\cdots[x+(m-1)h]},$$

$$\therefore \Delta u = \begin{cases} \dfrac{1}{(x+h)(x+2h)\cdots(x+mh)} \\ -\dfrac{1}{x(x+h)(x+2h)\cdots[x+(m-1)h]} \end{cases}$$

$$= \frac{-mh}{x(x+h)(x+2h)\cdots(x+mh)}.$$

Taking the integrals, and substituting for u its value, we have

$$\Sigma \frac{-1}{x(x+h)(x+2h)\cdots(x+mh)}$$

$$= \frac{1}{mhx(x+h)(x+2h)\cdots[x+(m-1)h]}.$$

In order that m may express the number of factors in the proposed difference, let it be changed into $m-1$, and the formula becomes

$$\Sigma \frac{1}{x(x+h)(x+2h)\cdots[x+(m-1)h]}$$

$$= \frac{-1}{(m-1)hx(x+h)(x+2h\cdots[x+(m-2)h]}\cdots[2].$$

(521.) Functions of the form

$$Ax^a + Bx^b + Cx^c \cdots$$

may without difficulty be integrated by the formula [1], which we have just obtained. For such functions may, in general, be transformed into products of equidifferent factors. As an example, let

$$x^3 = (x+h)(x+2h)(x+3h)$$
$$+ A(x+h)(x+2h) + B(x+h) + C,$$

h expressing the increment of x. This being developed and arranged by the powers of x, we find

$$\begin{aligned} x^3 = x^3 &+ 6hx^2 + 11h^2x + 6h^3 \\ &+ \text{A}x^2 + 3\text{A}hx + 2\text{A}h^2, \\ &+ \text{B}x + \text{B}h, \\ &+ \text{C}. \end{aligned}$$

That this equation should be identical, it is necessary that

$$\begin{aligned} 6h + \text{A} &= 0, \\ 11h^2 + 3\text{A}h + \text{B} &= 0, \\ 6h^3 + 2\text{A}h^2 + \text{B}h + \text{C} &= 0, \end{aligned}$$

which give

$$\text{A} = -6h, \quad \text{B} = 7h^2, \quad \text{C} = -h^3;$$

$$\therefore x^3 = (x+h)(x+2h)(x+3h) - 6h(x+h)(x+2h) + 7h^2(x+h) - h^3,$$

which by (519.), gives

$$\Sigma x^3 = \frac{1}{4h}x(x+h)(x+2h)(x+3h)$$

$$-2x(x+h)(x+2h) + \frac{7}{2}hx(x+h) - h^2x + \text{const.}$$

Since $\Sigma(-h^3) = -h^3\Sigma 1 = -h^2x$.

(522.) Each of the integrals

$$\Sigma x^0, \ \Sigma x, \ \Sigma x^2, \ \Sigma x^3, \ldots \Sigma x^{m-1}, \ \Sigma x^m,$$

depend on those which precede it, in such a manner, that if the $(m-1)$th be known, the mth may immediately be determined.

If each term of the equation (501.)

$$\Delta x^{m+1} = \frac{m+1}{1}x^m h + \frac{(m+1)m}{1.2}x^{m-1}h^2 + \frac{(m+1)m(m-1)}{1.2.3}x^{m-2}h^3$$

$$+ \frac{(m+1)m(m-1)(m-2)}{1.2.3.4}x^{m-3}h^4 \ldots + h^{m+1}x^0$$

be integrated, we obtain

$$x^{m+1}=\frac{m+1}{1}h\Sigma x^m+\frac{(m+1)m}{1.2}h^2\Sigma x^{m-1}$$
$$+\frac{(m+1)m(m-1)}{1.2.3}h^3\Sigma x^{m-2}\ldots\ldots h^{m+1}\Sigma x^0.$$

Hence we find

$$\Sigma x^m=\frac{x^{m+1}}{(m+1)h}-\left\{\frac{m}{1.2}h\Sigma x^{m-1}+\frac{m.m-1}{1.2.3}h^2\Sigma x^{m-2}\ldots\ldots\right.$$
$$\left.\ldots\ldots\frac{1}{m+1}h^m\Sigma x^0\right\}.$$

By the application of this formula, it is evident that by knowing the integral

$$\Sigma x^0=\frac{x}{h},$$

we may successively obtain

$$\Sigma x,\ \Sigma x^2,\ \Sigma x^3,\ldots\ldots\Sigma x^m,$$

by substituting successively $1, 2, 3, \ldots\ldots m$ for m.

Hence the results in the following table may easily be obtained:

$$\Sigma x^0=\frac{x}{h},$$

$$\Sigma x=\frac{x^2}{2h}-\frac{x}{2},$$

$$\Sigma x^2=\frac{x^3}{3h}-\frac{x^2}{2}+\frac{hx}{6},$$

$$\Sigma x^3=\frac{x^4}{4h}-\frac{x^3}{2}+\frac{hx^2}{4},$$

$$\Sigma x^4=\frac{x^5}{5h}-\frac{x^4}{2}+\frac{hx^3}{3}-\frac{h^3x}{30},$$

$$\Sigma x^5=\frac{x^6}{6h}-\frac{x^5}{2}+\frac{5hx^4}{12}-\frac{h^3x^2}{12},$$

$$\Sigma x^6=\frac{x^7}{7h}-\frac{x^6}{2}+\frac{hx^5}{2}-\frac{h^2x^3}{6}+\frac{h^5x}{42},$$

$$\Sigma x^7=\frac{x^8}{8h}-\frac{x^7}{2}+\frac{7hx^6}{12}-\frac{7h^3x^4}{24}+\frac{h^5x^2}{12},$$

$$\Sigma x^8 = \frac{x^9}{9h} - \frac{x^8}{2} + \frac{2hx^7}{3} - \frac{7h^3x^5}{15} + \frac{2h^5x^3}{9} - \frac{h^7x}{30},$$

$$\Sigma x^9 = \frac{x^{10}}{10h} - \frac{x^9}{2} + \frac{3hx^8}{4} - \frac{7h^3x^6}{10} + \frac{h^5x^4}{2} - \frac{3h^7x^2}{20}, \text{ \&c. \&c.}$$

In applying these formulæ to particular cases, the arbitrary constant should be supplied.

(523.) In general, let

$$\Sigma x^m = \text{A}x^{m+1} + \text{B}x^m + \text{C}x^{m-1} + \text{D}x^{m-2} \dots .$$

By taking the differences, we obtain

$$\begin{aligned} x^m = {} & \text{A}\frac{m+1}{1}x^m h \\ & + \text{A}\frac{(m+1)m}{1.2}x^{m-1}h^2 + \text{A}\frac{(m+1)m(m-1)}{1.2.3}x^{m-1}h^3 + \dots \\ & + \text{B}\frac{m}{1}x^{m-1}h + \text{B}\frac{m.(m-1)}{1.2}x^{m-2}h^2 + \dots \\ & \qquad + \text{C}\frac{m-1}{1}x^{m-2}h + \dots \\ & \qquad\qquad + \dots \end{aligned}$$

This will be rendered identical by the conditions

$$\text{A} = \frac{1}{m+1}h,$$

$$\text{B} = -\text{A}h\frac{m+1}{2} = -\tfrac{1}{2},$$

$$\text{C} = -\text{A}h^2\frac{(m+1)m}{2.3} - \text{B}h\frac{m}{2},$$

$$\text{D} = -\text{A}h^3\frac{(m+1)m(m-1)}{2.3.4} - \text{B}h^2\frac{m(m-1)}{2.3} - \text{C}h\frac{m-1}{2}$$

. . . .

Hence, we find in general

$$\begin{aligned} \Sigma x^m = {} & \frac{x^{m+1}}{(m+1)h} - \tfrac{1}{2}x^m, \\ & + \frac{1}{4.3}\ \frac{m}{1}hx^{m-1} - \frac{1}{6.5.4}\ \frac{m(m-1)(m-2)}{1.2.3}x^{m-3}h^3, \\ & + \frac{1}{36.7}\ \frac{m(m-1)(m-2)(m-3)(m-4)}{1.2.3.4.5}x^{m-5}h^5, \end{aligned}$$

$$-\frac{3}{10.9.8}\,\frac{m(m-1\cdots(m-6)}{1.2.3\cdots\cdots 7}x^{m-7}h^7,$$
$$+\frac{5}{60.11}\,\frac{m(m-1)\cdots(m-8)}{1.2.3\cdots\cdots 9}x^{m-9}h^9,$$
$$-\frac{691}{210.13.12}\,\frac{m(m-1)\ldots(m-10)}{1.2.3\ldots\ldots 11}x^{m-11}h^{11},$$
$$+\frac{35}{2.14.15}\,\frac{m(m-1)\ldots(m-12)}{1.2.3\ldots\ldots 13}x^{m-13}h^{13},$$
$$-\frac{3617}{30.17.16}\,\frac{m(m-1)\ldots(m-14)}{1.2.3\ldots\ldots 15}x^{m-15}h^{15},$$
$$+\frac{43867}{42.19.17}\,\frac{m(m-1)\ldots(m-16)}{1.2.3\ldots\ldots 17}x^{m-17}h^{17},$$
$$-\frac{1222277}{110.21.20}\,\frac{m(m-1)\ldots(m-18)}{1.2.3\ldots\ldots 19}x^{m-19}h^{19}.$$

. . . .

In this series, after the first two terms

$$\frac{x^{m+1}}{(m+1)h}-\tfrac{1}{2}x^m;$$

the succeeding terms may be found by multiplying the even terms (2nd, 4th, 6th . . . .) of the development of $(x+h)^m$ successively by the numeral factors

$$+\frac{1}{4.3},\quad -\frac{1}{6.5.4},\quad +\frac{1}{3.7.6},$$
$$-\frac{3}{10.9.8},\quad +\frac{5}{60.11},\ \&c.$$

These numeral coefficients are called the *numbers of Bernoulli*, because they were first determined by *James Bernoulli* *. They frequently occur in the theory of series.

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* For a full development of the properties of these remarkable numbers, see Mr. Herschel's excellent Treatise on Differences, with the examples on it.

In obtaining the above development in this way, the last term h^m of the development of $(x + h)^m$ should be omitted, even when it holds an even place.

(524.) To determine the method of integrating exponential functions, let $u_x = a^x$. Taking the difference, we find

$$\Delta u_x = a^x(a^h - 1),$$
$$\therefore u_x = \Sigma a^x(a^h - 1) = a^x,$$
$$\therefore \Sigma a^x = \frac{a^x}{a^h - 1}\text{*}.$$

Hence the method of integrating an exponential function.

(525.) Let $u_x = \cos.x$, $\therefore$

$\Delta \cos.x = \cos.(x + h) - \cos.x = - 2\sin.(x + \frac{1}{2}h)\sin.\frac{1}{2}h.$

Integrating this, we find

$$\Sigma \sin.(x + \tfrac{1}{2}h) = - \frac{\cos.x}{2\sin.\frac{1}{2}h},$$

or,

$$\Sigma \sin.y = - \frac{\cos.(y - \frac{1}{2}h)}{2\sin.\frac{1}{2}h},$$

by substituting y for $x + \frac{1}{2}h$.

By a similar process we obtain

$$\Sigma \cos.y = \frac{\sin.(y - \frac{1}{2}h)}{2\sin.\frac{1}{2}h}.$$

Powers of the sine and cosine are integrated by developing them in a series of sines or cosines of the multiples of the arc (Trigonometry), and then integrating the several terms of the development.

(526.) If the integral of the product of two functions x', x'', of x, be expressed thus,

$$\Sigma \mathrm{x}'\mathrm{x}'' = \mathrm{x}'\Sigma\mathrm{x}'' + \mathrm{x},$$

where x is an unknown function of x, let x be changed into

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* It may in general be observed that an arbitrary constant should be added in these integrations.

$x+h$ in $x'\Sigma x''+x$, and let x', x'' and x become $x'+\Delta x'$, $x''+\Delta x''$, and $x+\Delta x$ ∴

$$o=\Delta x'.\Sigma(x''+\Delta x'')+\Delta x,$$

$$\therefore x=-\Sigma[\Delta x'.\Sigma(x''+\Delta x'')],$$

$$\therefore \Sigma x'x''=x'\Sigma x''-\Sigma[\Delta x'.\Sigma(x''+\Delta x'')]$$

This formula corresponds to that found in the integral calculus for integration by parts.

(527.) The integral of a function considered as a difference can seldom be found in finite terms. Its value, however, may generally be expressed by a series. By Taylor's theorem,

$$\Delta z=\frac{dz}{dx}\cdot\frac{h}{1}+\frac{d^2z}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3z}{dx^3}\cdot\frac{h^3}{1.2.3}\cdots$$

Taking the integral of both members, we have

$$z=\frac{h}{1}\Sigma\frac{dz}{dx}+\frac{h^2}{1.2}\Sigma\frac{d^2z}{dx^2}+\frac{h^3}{1.2.3}\Sigma\frac{d^3z}{dx^3}+\cdots$$

If $u=\dfrac{dz}{dx}$ ∴ $z=\int udx$ ∴

$$\int udx=h\Sigma u+\alpha h^2\Sigma\frac{du}{dx}+\beta h^3\Sigma\frac{d^2u}{dx^2}+\cdots$$

where $\alpha,\beta,\gamma,\cdots$ represent the successive numerical coefficients. Hence we infer,

$$\Sigma u=\frac{1}{h}\int udx-\alpha h\Sigma\frac{du}{dx}-\beta h^2\Sigma\frac{d^2u}{dx^2}-\cdots$$

Taking the differential coefficients of each member, observing that

$$\frac{d\Sigma u}{dx}=\Sigma\frac{du}{dx}$$

we obtain

$$\Sigma\frac{du}{dx}=\frac{1}{h}u-\alpha h\Sigma\frac{d^2u}{dx^2}-\beta h^2\Sigma\frac{d^3u}{dx^3}-\cdots$$

$$\Sigma\frac{d^2u}{dx^2}=\frac{1}{h}\cdot\frac{du}{dx}-\alpha h\Sigma\frac{d^3u}{dx^3}-\beta h^2\Sigma\frac{d^4u}{dx^4}\cdots$$

$$\Sigma\frac{d^3u}{dx^3}=\frac{1}{h}\cdot\frac{d^2u}{dx^2}-\alpha h\Sigma\frac{d^4u}{dx^4}-\beta h^2\Sigma\frac{d^5u}{dx^5}\cdots$$

Eliminating successively the functions

$$\Sigma \frac{du}{dx}, \ \Sigma \frac{d^2u}{dx^2}, \ \&c. \ldots .$$

the final result must have the form

$$\Sigma u = \frac{1}{h}\int u dx + \text{A}u + \text{B}h \frac{du}{dx} + \text{C}h^2 \frac{d^2u}{dx^2} \ldots .$$

In a similar way we may obtain the values of the integrals $\Sigma\Sigma u$ or $\Sigma^2 u$, $\Sigma\Sigma\Sigma u$ or $\Sigma^3 u$, and in general for $\Sigma^m u$. The formula

$$\Delta^m z = \frac{d^m z}{dx^m} h^m + \alpha \frac{d^{m+1}z}{dx^{m+1}} h^{m+1} + \beta \frac{d^{m+2}z}{dx^{m+2}} h^{m+2} \ldots .$$

$$\therefore z = h^m \Sigma^m \frac{d^m z}{dx^m} + \alpha h^{m+1} \Sigma^m \frac{d^{m+1}z}{dx^{m+1}} + \beta h^{m+2} \Sigma^{m+2} \frac{d^{m+2}z}{dx^{m+2}} \ldots .$$

Let $\frac{d^m z}{dx^m} = u \therefore z = \int^m u dx^m \therefore$.

$$\Sigma^m u = \frac{1}{h^m} \int^m u dx^m - \alpha h \Sigma^m \frac{du}{dx} - \beta h^2 \Sigma^m \frac{d^2u}{dx^2} \ldots .$$

Assuming the differential coefficients of each member of this equation we find successively,

$$\Sigma^m \frac{du}{dx} = \frac{1}{h^m} \int^{m-1} u dx^{m-1} - \alpha h \Sigma^m \frac{d^2u}{dx^2} - \beta h^2 \Sigma^m \frac{d^3u}{dx^3} - \&c. \ldots$$

$$\Sigma^m \frac{d^2u}{dx^2} = \frac{1}{h^m} \int^{m-1} u dx^{m-2} - \alpha h \Sigma^m \frac{d^3u}{dx^3} - \beta h^2 \Sigma^m \frac{d^4u}{dx^4} - \&c. \ldots$$

. . . .

. . . .

Eliminating the functions

$$\Sigma^m \frac{du}{dx}, \ \Sigma^m \frac{d^2u}{dx^2}, \ \ldots .$$

the final result will have the form

$$\Sigma^m u = \frac{1}{h^m} \int^m u dx^m + \frac{\text{A}}{h^{m-1}} \int^{m-1} u dx^{m-1}$$

$$+ \frac{\text{B}}{h^{m-2}} \int^{m-2} u dx^{m-2} \ldots . + \frac{\text{M}}{h} \int u dx$$

$$+ \text{N}u + \text{P}h \frac{du}{dx} + \text{Q}h^2 \frac{d^2u}{dx^2} + \ldots .$$

SECTION V.

Of the summation of series.

(528.) If the successive terms of a series be expressed by the notation explained in (492.) the sum of all the terms from that whose *index* is 1 to that whose index is x inclusive may be expressed by $\text{s}u_x$; thus

$$\text{s}u_x = u_1 + u_2 \cdot \cdot \cdot \cdot \cdot \cdot + u_x.$$

In like manner

$$\text{s}u_{x+n} = u_1 + u_2 \cdot \cdot \cdot \cdot \cdot \cdot + u_x +_{x+1} + \cdot \cdot \cdot \cdot \cdot \cdot + u_{x+n}.$$

Subtracting the former from this we have

$$\text{s}u_{x+n} - \text{s}u_x = u_{x+1} + u_{x+2} + \cdot \cdot \cdot \cdot \cdot u_{x+n},$$

by which we may express the sum of any number of terms of a series commencing and terminating at any proposed terms.

(529.) If $n = 1$ we have

$$\text{s}u_{x+1} - \text{s}u_x = u_{x+1}.$$

But by (494.)

$$\Delta(\text{s}u_x) = \text{s}u_{x+1} - \text{s}u_x,$$

$$\therefore \Delta \text{s}u_x = u_{x+1},$$

$$\therefore \text{s}u_x = \Sigma u_{x+1} + \text{c},$$

c being the arbitrary constant. When $x = 0$, $\text{s}u_x = 0$ $\therefore$

$$0 = \Sigma u_1 + \text{c}.$$

Subtracting this equation from the last we find

$$\text{s}u_x = \Sigma u_{x+1} - \Sigma u_1.$$

Hence the summation of the series depends on the integration of u_{x+1} and u_1 considered as differences.

(530.) In like manner if the sum of the series from the nth to the xth term, including the latter, be required, we have

$$su_x - su_n = u_{x-1} + u_{x+2} \cdot \cdot \cdot \cdot \cdot + u_n.$$

But by what has just been proved,

$$su_x = \Sigma u_{x+1} - \Sigma u_1,$$

$$su_n = \Sigma u_{n+1} - \Sigma u_1,$$

$$\therefore su_x - su_n = \Sigma u_{x+1} - \Sigma u_{n+1}.$$

(531.) We shall now give some examples of the application of these principles to the summation of series.

Ex. 1. *To determine the sum of a series of figurate numbers of the first, second, and successive orders, beginning with unity in each series.*

The figurates of the first order are the series of integers

$$1, 2, 3, 4, \cdot \cdot \cdot \cdot$$

of which the general term is x.

Those of the second order are

$$\frac{1.2}{1.2}, \frac{2.3}{1.2}, \frac{3.4}{1.2}, \frac{4.5}{1.2}, \cdot \cdot \cdot \cdot$$

of which the general term is $\frac{x.\overline{x+1}}{1.2}$.

Those of the third order are

$$\frac{1.2.3}{1.2.3}, \frac{2.3.4}{1.2.3}, \frac{3.4.5}{1.2.3}, \cdot \cdot \cdot \cdot$$

of which the general term is $\frac{x.\overline{x+1}.\overline{x+2}}{1.2.3}$.

Those of the fourth order are

$$\frac{1.2.3.4}{1.2.3.4}, \frac{2.3.4.5}{1.2.3.4}, \frac{3.4.5.6}{1.2.3.4}, \cdot \cdot \cdot \cdot$$

of which the general term is $\frac{x.\overline{x+1}.\overline{x+2}.\overline{x+3}}{1.2.3.4}$.

And in general the figurates of the nth order are

$$\frac{1.2.3 \cdot \cdot \cdot n}{1.2.3 \cdot \cdot \cdot n}, \frac{2.3.4 \cdot \cdot \cdot \overline{n+1}}{1.2.3 \cdot \cdot \cdot n}$$

$$\frac{3.4.5 \cdot \cdot \cdot \overline{n+2}}{1.2.3 \cdot \cdot \cdot n}, \frac{4.5.6 \cdot \cdot \cdot \cdot \overline{n+3}}{1.2.3 \cdot \cdot \cdot \cdot n}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

of which the general term is

$$\frac{x.x+1.x+2\ldots\ldots\ldots x+(n-1)}{1.2.3\ldots\ldots\ldots n}.$$

For those of the first order we have

$$u_x = x.$$

Hence by (529.)

$$\mathrm{s}u_x = \Sigma(x+1) + \mathrm{c}.$$

By the table in (522.) we have

$$\Sigma x = \frac{x^2}{2h} - \frac{x}{2}.$$

Changing x into $x+1$ and h into 1, we have

$$\Sigma u_{x+1} = \frac{(x+1)^2 - x}{2} + \mathrm{c}$$

$$\therefore \Sigma u_1 = \tfrac{1}{2} + \mathrm{c}.$$

Subtracting this from the former, we have

$$\Sigma u_{x+1} - \Sigma u_1 = \frac{x(x+1)}{1.2} = \mathrm{s}u_x$$

$$\therefore 1+2+3\cdots\cdots+x = \frac{x.x+1}{1.2}.$$

For the figurates of the second order,

$$u_x = \frac{x.x+1}{1.2} \quad \therefore u_{x+1} = \frac{x+1.x+2}{1.2}.$$

By the formula established in (519.),

$$\Sigma u_{x+1} = \frac{x.x+1.x+2}{1.2.3} + \mathrm{c}.$$

Hence

$$\mathrm{s}u_x = \frac{x\cdot x+1\cdot x+2}{1\cdot 2\cdot 3}$$

no constant being added because when $x=0$, $\mathrm{s}u_x = 0$.

In general for the sum of the figurates of the nth order

$$u_{x+1} = \frac{x+1\cdot x+2\cdot x+3\cdots\cdots x+n}{1\cdot 2\cdot 3\cdots\cdots n}.$$

$$\therefore \Sigma u_{x+1} = \frac{x\cdot x+1\cdot x+2\cdots\cdots x+n}{1\cdot 2\cdot 3\cdots\cdots n+1}.$$

$$\because \text{s}u_x = \frac{x \cdot x+1 \cdot x+2 \cdot \cdots \cdot x+n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n+1}.$$

Ex. 2. *To determine the sum of a series of the reciprocals of the figurate numbers beginning from unity.*

The general terms of the several series are in this case,

$$1^{\circ}. \quad \frac{1}{x},$$

$$2^{\circ}. \quad \frac{1 \cdot 2}{x(x+1)},$$

$$3^{\circ}. \quad \frac{1 \cdot 2 \cdot 3}{x(x+1)(x+2)}, \text{ \&c.}$$

Hence, by (520.) the sums for 2°, 3°, &c.

$$\text{s}\,\frac{1 \,.\, 2}{x(x+1)} = -\frac{2}{x+1} + \text{c},$$

$$\text{s}\,\frac{1 \cdot 2 \cdot 3}{x(x+1)(x+2)} = -\frac{3}{(x+1)(x+2)} + \text{c, \&c.}$$

The formula (520.) fails for $\text{s}\left(\frac{1}{x}\right)$. The constants being supplied by the condition

$$0 = -\frac{2}{1} + \text{c},$$

$$0 = -\frac{3}{1 \cdot 2} + \text{c, \&c.}$$

give

$$\text{s}\,\frac{1 \cdot 2}{x(x+1)} = \frac{2}{1} - \frac{2}{x+1} = \frac{2x}{x+1},$$

$$\text{s}\,\frac{1 \cdot 2 \cdot 3}{x(x+1)(x+2)} = \frac{3}{1 \,.\, 2} - \frac{3}{(x+1)(x+2)}, \text{ \&c.}$$

When x is infinite, these values become $\frac{2}{1}$, $\frac{3}{1 \cdot 2}$, &c.

Ex. 3. To determine the sum of

$$1^3 + 2^3 + 3^3 \cdots\cdots x^3.$$

By the table in (522.) we have

$$\Sigma\,(x+1)^3 = \frac{(x+1)^4}{4} - \frac{(x+1)^3}{2} + \frac{(x+1)^2}{4}$$

$$=\left(\frac{x(x+1)}{2}\right)^2$$

$$\therefore \text{s}u_x = \left(\frac{x(x+1)}{2}\right)^2.$$

Hence it follows that

$$1^3 + 2^3 + 3^3 \cdots x^3 = (1 + 2 + 3 \cdots x)^2.$$

Ex. 4. To determine the sum of an arithmetical series

$$a + (a + d) + (a + 2d) \cdots\cdots + [a + (x - 1)d].$$

In this case

$$u_x = a + (x - 1)d,$$

$$\therefore u_{x+1} = a + xd,$$

$$\therefore \Sigma u_{x+1} = xa + \frac{x(x-1)}{1 \cdot 2} d.$$

Hence

$$\text{s}u_x = xa + \frac{x \cdot x-1}{1 \cdot 2} d.$$

Ex. 5. To find the sum of a geometrical series

$$a, ar, ar^2, \cdots\cdots ar^{x-1}.$$

In this case

$$u_x = ar^{x-1}, \therefore u_{x+1} = ar^x.$$

Hence (524.),

$$\Sigma u_{x+1} = a\frac{r^x}{r-1} + \text{C},$$

$$\therefore \text{s}u_x = \frac{ar^x}{r-1} + \text{C}.$$

When $x = 0$, $\text{s}u_x = 0$, $\therefore$

$$0 = \frac{a}{r-1} + \text{C},$$

$$\therefore \text{C} = -\frac{a}{r-1},$$

$$\therefore \text{s}u_x = \frac{a(r^x - 1)}{r-1}.$$

Ex. 6. To find the sum of the series

$$\text{s}u_x = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} \cdots + \frac{1}{x(x+1)}.$$

In this case (520.),

$$\Sigma u_{x+1} = \Sigma \frac{1}{(x+1)(x+2)} = c - \frac{1}{x+1}.$$

Whence c = 1, ∵

$$\mathrm{s}u_x = \frac{x}{x+1}.$$

Ex. 7. To determine the sum of the series

$$1^2 + 2^2 + 3^2 \ldots\ldots + x^2.$$

In this case,

$$u_{x+1} = (x+1)^2,$$

$$\Sigma u_{x+1} = \frac{(x+1)^3}{3} - \frac{(x+1)^2}{2} + \frac{(x+1)}{6} + c,$$

$$\because \Sigma u_1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{6} + c,$$

$$\because \mathrm{s}u_x = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}.$$

Ex. 8. To determine the sum of the series

$$\mathrm{s}u_x = \cos.\ \varphi + \cos.\ 2\varphi + \cos.\ 3\varphi \ldots\ldots + \cos.\ x\varphi.$$

Hence (525.),

$$u_{x+1} = \cos.\ (x+1)\varphi,$$

$$\Sigma u_{x+1} = \frac{\sin.\ (x+\frac{1}{2})\varphi}{2 \sin.\ \frac{1}{2}\varphi} + c,$$

$$\because \Sigma u_1 = \frac{\sin.\ \frac{1}{2}\varphi}{2 \sin.\ \frac{1}{2}\varphi} + c,$$

$$\because \mathrm{s}u_x = \Sigma u_{x+1} - \Sigma u_1 = \frac{\sin.(x+\frac{1}{2})\varphi - \sin.\ \frac{1}{2}\varphi}{2 \sin.\ \frac{1}{2}\varphi}.$$

Ex. 9. To find the sum of

$$\mathrm{s}u_x = \sin.\ \varphi + \sin.\ 2\varphi + \sin.\ 3\varphi \ldots\ldots \sin.\ x\varphi,$$

$$u_{x+1} = \sin.\ (x+1)\varphi,$$

$$\Sigma u_{x+1} = -\frac{\cos.\ (x+\frac{1}{2})\varphi}{2 \sin.\ \frac{1}{2}\varphi} + c,$$

$$\Sigma u_1 = -\frac{\cos.\ \frac{1}{2}\varphi}{2 \sin.\ \frac{1}{2}\varphi} + c,$$

$$\because \mathrm{s}u_x = \frac{\cos.\ \frac{1}{2}\varphi - \cos.\ (x+\frac{1}{2})\varphi}{2 \sin.\ \frac{1}{2}\varphi}.$$

Ex. 10. To sum the series

$$\text{s}u_x = 1 \cdot 3^2 + 3 \cdot 5^2 + 5 \cdot 7^2 + \ldots\ldots$$

In this case

$$u_x = (2x - 1)(2x + 1)^2,$$

$$\therefore u_{x+1} = (2x + 1)(2x + 3)^2 = 8x^3 + 28x^2 + 30x + 9.$$

By (516.), $\therefore$

$$\Sigma u_{x+1} = 8\Sigma x^3 + 28\Sigma x^2 + 30\Sigma x + 9\Sigma x^0.$$

By substituting the values in (520.), we find

$$\Sigma u_{x+1} = \frac{x(6x^3+16x^2+9x-4)}{3} + \text{c},$$

$$\therefore \Sigma u_1 = 0 + \text{c},$$

$$\therefore \text{s}u_x = \frac{x(6x^3+16x^2+9x-4)}{3}.$$

Ex. 11. To sum the series

$$\text{s}u_x = 1^2 + 3^2 + 5^2 \ldots\ldots$$

In this case

$$u_x = (2x-1)^2,$$

$$\therefore u_{x+1} = (2x + 1)^2 = 4x^2 + 4x + 1,$$

$$\therefore \Sigma u_{x+1} = \frac{x(4x^2-1)}{3} + \text{c},$$

$$\Sigma u_1 = 0 + \text{c},$$

$$\therefore \text{s}u_x = \frac{x(4x^2-1)}{3}.$$

THE END.

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